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★**Spherical harmonics and approximations on the unit sphere: an introduction.**

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Spherical harmonics have been studied extensively and applied to solving a wide range of problems in science and engineering. Interest in approximation and numerical analysis for problems over spheres has grown steadily in recent years. This book provides an introduction to the theory of spherical harmonics in an arbitrary dimension as well as an account of classical and recent results on some aspects of approximation theory and numerical integration over the sphere.

After a preliminary discussion in the first chapter, introducing in particular some necessary notations and basic results related to the sphere and the gamma functions, a detailed theory of spherical harmonics is developed in Chapter 2. Several important formulas and properties involving the Laplace-Beltrami operator and spherical harmonics are discussed and used to study the Sobolev spaces on the sphere in Chapter 3. After that, in Chapter 4, the authors present several interesting results on best approximation by spherical polynomials and their analogues for polynomial approximation on the unit disk. In particular, moduli of smoothness and direct and inverse theorems for polynomial approximation are discussed in this chapter. Various aspects of cubature formulas on the sphere are discussed in Chapter 5. Finally, in Chapter 6, the authors discuss how to apply the theory of spherical harmonics to solve the Dirichlet problem

$$\begin{aligned} -\Delta u(x) &= 0, \quad x \in \Omega \subset \mathbb{R}^3, \\ u(x) &= f(x), \quad x \in \partial\Omega, \end{aligned}$$

and the Neumann problem

$$\begin{aligned} -\Delta u(x) + \gamma(x) u(x) &= f(x), \quad x \in \Omega \subset \mathbb{R}^2, \\ \frac{\partial u(x)}{\partial \nu_x} &= g(x), \quad x \in \partial\Omega, \end{aligned}$$

where Ω is an open simply connected region having a smooth boundary, and they give a Galerkin method for the Beltrami-type equation

$$-\Delta^* u + c_0 u = f \ \text{ on } \ \mathbb{S}^{d-1},$$

where $c_0 > 0$ and f.

This is a very well-written, self-contained monograph on spherical harmonics. It is an excellent reference source for researchers and graduate students who are interested in polynomial approximation, numerical integration, differentiation and solution of partial differential and integral equations over the sphere.

Reviewed by *Feng Dai*