APX-Hardness of Domination Problems in Circle Graphs

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Abstract

We show that the problem of finding a minimum dominating set in a circle graph is APX-hard: there is a constant $\delta > 0$ such that, there is no $(1+\delta)$-approximation algorithm for the minimum dominating set problem on circle graphs unless P = NP. Hence a PTAS for this problem seems unlikely. This hardness result complements the $(2+\varepsilon)$-approximation algorithm for the problem (*Journal of Algorithms*, 42(2), 255-276, 2002).

1 Introduction

A graph $G = (V, E)$ is a *circle graph* if there is a one-to-one correspondence between vertices in V and a set C of chords in a circle such that two vertices in V are adjacent if and only if the corresponding chords in C intersect. A subset V' of V is a *dominating set* of G if for all $u \in V$ either $u \in V'$ or u has a neighbor in V'. Keil [4] showed that the problem of finding a minimum cardinality dominating set (MDS) is NP-complete for circle graphs. In this paper we study the inapproximability of MDS. In this paper we study the inapproximability of MDS.

The class *APX* is the class of optimization problems, each of which has an α-approximation algorithm for some constant α. A *polynomial time approximation scheme (PTAS)* is a family F of approximation algorithms such that for each $\varepsilon > 0$, there is a $(1+\varepsilon)$-approximation algorithm A_ε in F with running time polynomial in the input size. An optimization problem is said to be *APX-hard* if a PTAS for the problem implies that *every* problem in APX has a PTAS. Furthermore, as shown by Arora et al. [1], in this case $P = NP$.

In this paper we show that MDS is APX-hard. This is shown via a *gap-preserving* reduction [5] from an optimization version of the 3-SAT problem

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called *MAX-3SAT(8)* (defined in Section 2). This APX-hardness results complements the $(2+\varepsilon)$-approximation algorithm for MDS on circle graphs presented in [3].

2 APX-hardness of MDS

Our results are based on a "gap-preserving" reduction from MAX-3SAT(8), that uses ideas in [4] in which the NP-completeness of MDS is established. The problem MAX-3SAT(k) is defined below.

MAX-3SAT(k)
`INPUT`: A set $X = \{x_1, x_2, \ldots, x_n\}$ of variables and a set $C = \{c_1, c_2, \ldots, c_m\}$ of disjunctive clauses such that each clause contains at most 3 literals and each variable occurs in at most k clauses.
`OUTPUT`: A truth-assignment to variables in X that maximizes the number of clauses in C satisfied.

For any instance ϕ of MAX-3SAT(k), let SAT(ϕ) denote the largest fraction of clauses in ϕ that can be simultaneously satisfied. For any graph G let $\gamma(G)$ denote the size of a minimum dominating set in G. We show the following theorem.

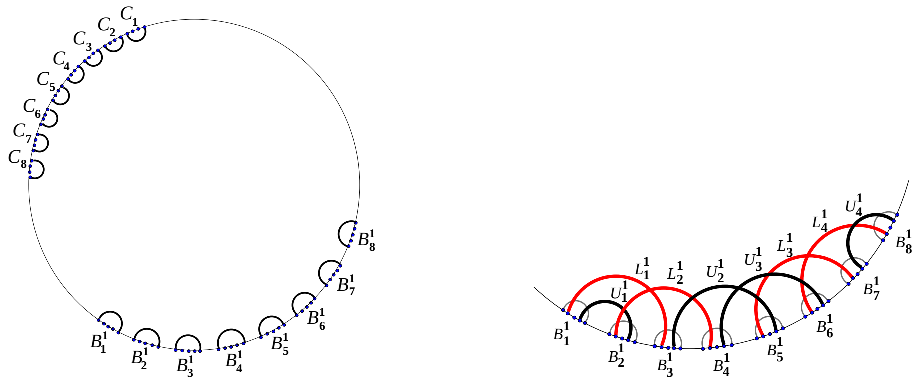
Theorem 1 *There is a polynomial time reduction that takes an instance ϕ of MAX-3SAT(8) with n variables and m clauses and constructs a circle graph G such that*

$$SAT(\phi) = 1 \Rightarrow \gamma(G) \le 16n + 2$$
$$SAT(\phi) < \alpha \Rightarrow \gamma(G) > 16n + 2 + (1-\alpha)m/8$$

2.1 The reduction

Let ϕ be an instance of MAX-3SAT(8); without loss of generality we assume that each variable appears in exactly 8 clauses. We now show a polynomial-time reduction that maps ϕ to a circle graph such that the above theorem holds. We construct in polynomial time from ϕ, a set J of chords of a circle. The theorem holds for the circle graph $G(J)$ induced by the chords in J. Since this reduction is similar to the reduction in [4] (Theorem 2.1) we do not present details such as co-ordinates of endpoints of chords, merely emphasizing intersections between chords. As a running example for the reduction we consider an instance ϕ of MAX-3SAT(8) in which the literal x_1 appears in clauses c_1, c_2, c_4, and $\overline{x}_1$ appears in c_3, c_5, c_6, c_7, c_8.

The set J contains m pairwise non-intersecting chords $C_1, C_2, \ldots, C_m$ corresponding respectively to the clauses $c_1, c_2, \ldots, c_m$. The chords C_1, C_2, …, C_m are placed in counterclockwise order around the circle as shown in Figure 1(a). For each variable x_i and each clause c_j, the set J contains a *base chord* B^i_j,



(a) (b)

Fig. 1. $m = 8$ (a) Clause chords and base chords corresponding to variable x_1 (b) The lower and and upper chords corresponding to variable x_1.

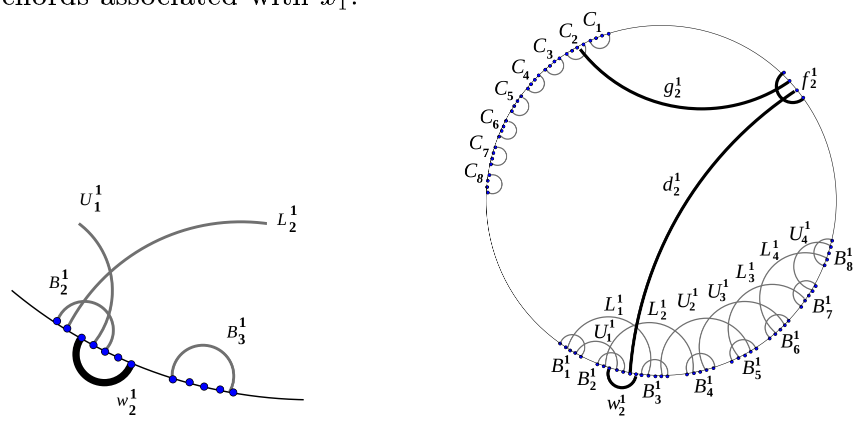
provided x_i appears (as x_i or $\overline{x}_i$) in clause c_j. Thus the number of base chords in J associated with each x_i is exactly 8. This step of the reduction differs from Keil's reduction in which exactly m base chords corresponded to each variable x_i, independent of the number of clauses x_i appeared in. The base chords are pairwise non-intersecting and are placed as in Figure 1(a). Specifically, as we travel counterclockwise around the circle starting from any clause chord, we first encounter the base chords for x_1, then the base chords for x_2, and so on. Assuming that variable x_i appears in clauses c_{j_t}, with $1 \leq t \leq 8$ and $j_1 < j_2 < \cdots < j_8$, then the base chords $B^i_{j_1}, B^i_{j_2}, \ldots, B^i_{j_8}$ appear in this order as we travel counterclockwise around the circle.

For each variable x_i, we add to J four *upper chords* $U^i = \{U^i_1, U^i_2, U^i_3, U^i_4\}$ and four *lower chords* $L^i = \{L^i_1, L^i_2, L^i_3, L^i_4\}$. Each chord in $U^i \cup L^i$ intersects exactly two base chords corresponding to x_i. Suppose variable x_i appears in clauses c_{j_t}, $1 \leq t \leq 8$, $j_1 < j_2 < \cdots < j_8$. Then U^i_1 intersects $B^i_{j_1}$ and $B^i_{j_2}$; U^i_2 intersects $B^i_{j_3}$ and $B^i_{j_5}$; U^i_3 intersects $B^i_{j_4}$ and $B^i_{j_6}$; and U^i_4 intersects $B^i_{j_7}$ and $B^i_{j_8}$. The lower chords intersect the base chords as follows: L^i_1 intersects $B^i_{j_1}$ and $B^i_{j_3}$; L^i_2 intersects $B^i_{j_2}$ and $B^i_{j_4}$; L^i_3 intersects $B^i_{j_5}$ and $B^i_{j_7}$; and L^i_4 intersects $B^i_{j_6}$ and $B^i_{j_8}$. Figure 1(b) shows the placement of the upper and lower chords for variable x_1 and how their interaction with the base chords.

Note that chords in U^i dominate all base chords corresponding to x_i and similarly chords in L^i dominate all base chords corresponding to x_i. For any dominating set D of $G(J)$, $U^i \subseteq D$ and $L^i \cap D = \emptyset$ corresponds to setting x_i true, and $L^i \subseteq D$ and $U^i \cap D = \emptyset$ corresponds to setting x_i false.

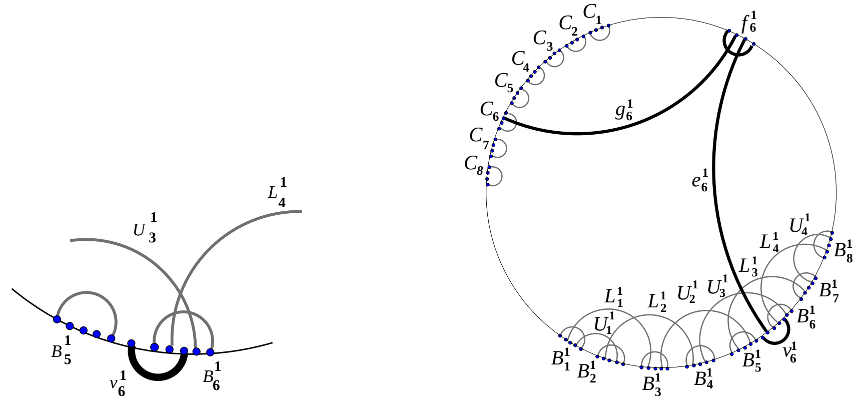
We include in J four more chords associated with each variable x_i that appears in a clause c_j. If the literal x_i appears in C_j then we add the chords w^i_j, d^i_j, f^i_j and g^i_j to J. These chords induce a simple path from w^i_j to C_j in $G(J)$. The chord w^i_j intersects B^i_j and an upper chord in U^i. See Figure 2(a). Thus in any set containing all the chords in U^i, w^i_j is dominated. Dominating C_j

with g_j^i corresponds to satisfying c_j by setting x_i to true. Figure 2(b) shows the chords w_2^1, d_2^1, f_2^1 and g_2^1. Figure 4(a) shows all type w, type d, type f and type g chords associated with x_1.



(a) (b)

Fig. 2. x_1 appears in C_2: J contains a sequence of chords $w_2^1, d_2^1, f_2^1, g_2^1$ that induce a path from w_2^1 to C_2. (a) Placement of w_2^1 (b) Placement of d_2^1, f_2^1, and g_2^1.

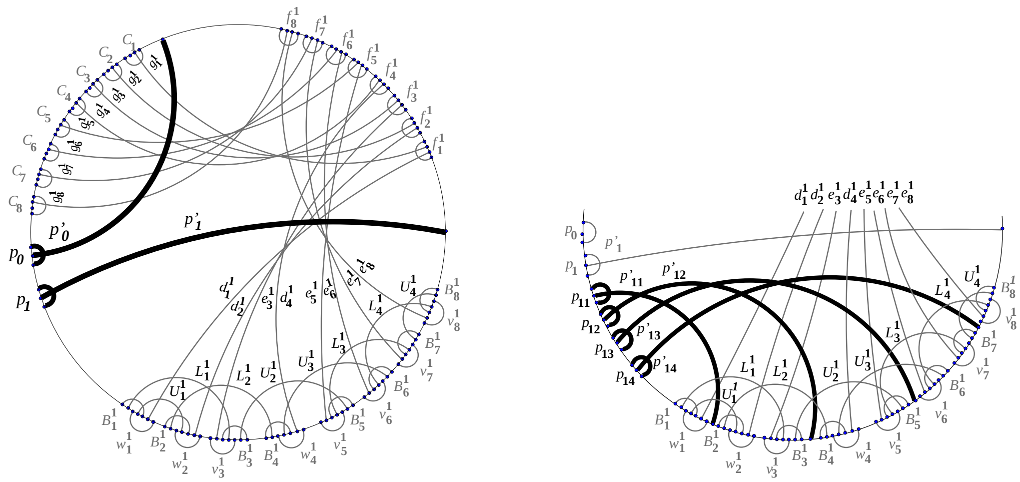


(a) (b)

Fig. 3. $\overline{x}_1$ appears in C_6: J contains a sequence of chords $v_6^1, e_6^1, f_6^1, g_6^1$ that induce a path from v_6^1 to C_6. (a) Placement of v_6^1 (b) Placement of v_6^1, e_6^1, f_6^1 and g_6^1.

If $\overline{x}_i$ appears in C_j, then we include in J four chords v_j^i, e_j^i, f_j^i and g_j^i. Again, these chords induce a simple path in the circle graph from v_j^i to C_j. The chord v_j^i intersects B_j^i and a lower chord in L^i. See Figure 3(a). Thus in any dominating set containing all the chords in L^i, v_j^i is dominated. As before, dominating C_j with g_j^i corresponds to satisfying C_j with x_i, but by setting x_i to false. Figure 3(b) shows chords v_6^1, e_6^1, f_6^1 and g_6^1. Figure 4(a) shows all type v, type e, type f and type g chords associated with x_1.

All type f chords are grouped together as in Figure 4(a). As we travel counterclockwise from the clause chords, we first encounter all the base chords, then the 8 type f^1 chords, followed by the 8 type f^2 chords, and so on. If a variable x_i appears in clauses c_{j_t}, $1 \le t \le 8$, $j_1 < j_2 < \cdots < j_8$, then the 8 type f^i chords $f^i_{j_1}, f^i_{j_2}, \ldots, f^i_{j_8}$ appear in this order counterclockwise around the circle. Next we add a pair of chords p'_1 and p_1 so that p'_1 intersects all the type d and e chords and p_1 intersects only p'_1 (see Figure 4(a)). This implies that if D is a dominating set of $G(J)$ that does not contain p'_1, there exists a dominating set, no larger, than contains p'_1. Including p'_1 in a dominating set D will enable us to treat all the type d and type e chords as dominated; such chords will occur in D only if they are needed to dominate *other* chords. Similarly, we add a pair of chords p'_0 and p_0 such that p'_0 intersects all type g chords and p_0 intersects exactly p'_0 (see Figure 4(a)). Again, there exists a minimum dominating set that contains p'_0.



(a) (b)

Fig. 4. (a) p'_0 dominates all type g type intervals; p'_1 dominates all type d and e intervals (b) type p intervals dominate all type U and L intervals: p'_{11} dominates U^1_1, L^1_1; p'_{12} dominates U^1_2, L^1_2; p'_{13} dominates U^1_3, L^1_3; and p'_{14} dominates U^1_4, L^1_4.

Finally, we add to J $4n$ pairs of chords: p_{is} and p'_{is}, $1 \le i \le n$, $1 \le s \le 4$. Each p_{is} intersects exactly one chord, p'_{is}. For each i, $1 \le i \le n$, the chords in $\{p'_{is} \mid 1 \le s \le 4\}$ collectively dominate all the type U^i and type L^i chords, as shown in Figure 4(b). None of the p_{is} chords intersectx a base chord of x_i. This completes our construction; see [4] for details such as actual coordinates of endpoints.

The total number of vertices in our circle graph is $|J| = m + 56n + 4$. To see this, observe that J contains m clause chords. For each variable x_i, J contains: 8 base chords, 4 upper chords, 4 lower chords, and 32 chords of types w, v, d, e, f and g. Thus for each variable we have 48 chords. In addition, we have a total of $4 + 8n$ type p and p' chords for a total of $m + 56n + 4$ chords.

2.2 Analysis

Lemma 2 *SAT*$(\phi) = 1 \Rightarrow \gamma(G) \leq 16n + 2$.

PROOF. Since SAT$(\phi) = 1$, there is a satisfying truth assignment A for ϕ. We construct a dominating set D of size $16n + 2$ using the procedure described in [4], which we briefly sketch here. As mentioned earlier, we can assume without loss of generality that D contains all the type $p^{'}$ chords. There are $4n + 2$ such chords and they dominate all the type p, U, L, d, e and g chords. It remains to dominate the type B, C, v, w and f chords. If x_i is true in A, we include in D all type U^i chords; if x_i is false in A, we include all L^i chords. Thus we have added $4n$ more chords to D and have dominated all base chords.

Suppose that the literal x_i appears in a clause c_j. Then if x_i is true in A, w_j^i is dominated by chords in U^i, and d_j^i is dominated by a type p' chord. We add the chord g_j^i to D to dominate f_j^i. If x_i is false in A, we add d_j^i to dominate w_j^i and f_j^i. In either case, we use a single chord for the (i, j) pair. Now suppose that the literal $\overline{x}_i$ appears in a clause c_j. Then if x_i is false in A, v_j^i is dominated by chords in L^i, and e_j^i is dominated by a type p' chord. We add the chord g_j^i to D to dominate f_j^i. If x_i is false in A, we add e_j^i to dominate v_j^i and f_j^i. In either case, we add a single chord for the (i, j) pair.

Since there are $8n$ possible (i, j) pairs, we add $8n$ additional chords and dominate all the type v, type w, and type f chords. Since A is a satisfying truth assignment, every clause c_j is dominated by a chord g_j^i for some i. The number of chords we have included in D is $16n + 2$.

Lemma 3 *SAT*$(\phi) < \alpha \Rightarrow \gamma(G) > 16n + 2 + (1 - \alpha)m/8$.

PROOF. We prove this by showing that if $G(J)$ has a dominating set D of size $|D| \leq 16n+2+(1-\alpha)m/8$, then there is a truth assignment to variables in X that satisfies at least αm of the clauses. For any subset $J' \subseteq J$ of chords, define the *D-dominating set* of J' as

$$(J' \cup \{y \in J \mid y \text{ is a neighbor of some vertex in } J'\}) \cap D.$$

Let D_p, D_B, and D_f be D-dominating sets respectively for the set of type p chords, the set of base chords, and the set of type f chords. There are $(4n + 2)$ type p chords, no two of which have a common neighbor and so $|D_p| \geq (4n+2)$. There are $8n$ base chords, no three of which share a neighbor and so $|D_B| \geq 4n$. There are $8n$ type f chords, no two of which share a neighbor and so $|D_f| \geq 8n$. Also, the sets D_p, D_B, and D_f are pairwise non-intersecting because no chord in J intersects a type p chord and a base chord, or a base chord and a type f chord, or a type f chord and a type p chord. Therefore, the inequality $|D_p| \geq 4n+2$ implies that $|D_B| + |D_f| \leq 12n + (1-\alpha)m/8$. For notational convenience, let $\beta = 1-\alpha$ and let $\beta_1, \beta_2 \geq 0$ be reals such that $|D_B| = 4n + \beta_1 m/8$ and $|D_f| = 8n + \beta_2 m/8$. This implies that $\beta_1 + \beta_2 \leq \beta$.

For any i, $1 \leq i \leq n$, let $D_B^i \subseteq D_B$ be the D-dominating set for the base chords corresponding to variable x_i. It is easy to verify that $|D_B^i| \geq 4$ and if $|D_B^i| = 4$ then $D_B^i = U^i$ or $D_B^i = L^i$. Note that the sets D_B^i are pairwise disjoint for distinct i's and therefore $|D_B| = \sum_{i=1}^{n} |D_B^i|$. Since $|D_B| = 4n + \beta_1 m/8$, this implies that for at most $\beta_1 m/8$ of the i's we have $|D_B^i| > 4$, while for the rest of the i's we have $|D_B^i| = 4$ and therefore $D_B^i = U^i$ or $D_B^i = L^i$. For any i, $1 \leq i \leq n$, if $D_B^i = U^i$ assign to x_i the value true; otherwise if $D_B^i = L^i$, assign to x_i the value false. Variables to which truth values have been assigned are called *consistent*; the remaining variables are called *inconsistent*. Arbitrarily assign truth values to inconsistent variables. Next we show that this truth assignment satisfies at least αm of the clauses.

Note that there are at most $\beta_1 m/8$ inconsistent variables. These can participate in at most $\beta_1 m$ clauses and therefore the remaining at least $(1-\beta_1)m$ clauses contain only consistent variables. Let $C' \subseteq C$ denote the subset of clauses that contain only consistent variables. Call any clause in C' a *consistent clause* and call any type C chord that corresponds to a clause in C' a *consistent chord*. Let t be the number of consistent chords in D. It follows that $t \leq (\beta - \beta_1 - \beta_2)m/8$, because otherwise,

$$\begin{aligned} |D| &\geq |D_p| + |D_B| + |D_f| + t \\ &> (4n+2) + (4n + \beta_1 m/8) + (8n + \beta_2 m/8) + (\beta - \beta_1 - \beta_2)m/8 \\ &= 16n + 2 + \beta m/8 \end{aligned}$$

a contradiction. This implies that the number of consistent chords dominated by type g chords is at least

$$(1-\beta_1)m - (\beta - \beta_1 - \beta_2)\frac{m}{8} \geq (1-\beta)m + \frac{\beta_2 m}{8}.$$

Let S denote the set of indices j such that c_j is a consistent chord dominated by a type g chord. Now construct a set F of type f chords as follows: for each $j \in S$, pick a g_j^i that dominates c_j (we know such a g_j^i exists) and add the chord f_i^j to F. For each $j \in S$, F contains exactly one f_j^i for some i. Also note that $|F| \geq (1-\beta)m + \frac{\beta_2 m}{8}$ and all of the chords in F are dominated by type g chords. Of the chords in F, at most $\beta_2 m/8$ chords can be dominated by 2 or more chords. This is because $|D_f| = 8n + \beta_2 m/8$ and no two type f chords share a neighbor. Hence, there are at least $(1-\beta)m = \alpha m$ type f chords in F that are dominated *only* by type g chords.

Now consider a chord $f_j^i \in F$, dominated only by a type g chord. Since f_j^i is in J, either x_i or $\overline{x}_i$ appears in c_j. Suppose that x_i appears in c_j. Then we have the chords w_j^i, d_j^i, also in J. Since f_j^i is dominated only by type g chords,

$d^i_j \notin D$ and this in turn implies that either $w^i_j \in D$ or there is an upper chord that dominates it. Since C_j is a consistent chord, it only contains consistent variables and therefore $D^i_B = U^i$ or $D^i_B = L^i$. Hence, $w^i_j \notin D$, implying that $D^i_B = U^i$, which in turn implies that x_i is assigned true and therefore clause c_j is satisfied.

A similar argument suffices to show that c_j is satisfied even in the case when $\overline{x}_i$ appears in c_j. Therefore, the truth assignment satisfies at least αm clauses. This completes the proof.

For any instance of ϕ of MAX-3SAT(8) with n variables and m clauses, we can assume that $m \geq \frac{n}{3}$. As a consequence, Theorem 1 implies that if SAT$(\phi) < \alpha$ then

$$\gamma(G) > \left(16 + \frac{1-\alpha}{24}\right) n + 2.$$

Let $\beta = \frac{(16+(1-\alpha)/24)}{16}$ and let $\epsilon > 0$ be fixed. Suppose that there is $(\beta - \epsilon)$-approximation algorithm for MDS on circle graphs. Then there is a constant $n(\epsilon)$, such that for any $n > n(\epsilon)$ and for any instance ϕ of MAX-3SAT(8) with n variables, using the "gap-preserving" reduction in the proof of Theorem 1, it can be determined in polynomial time whether $SAT(\phi) = 1$ or $SAT(\phi) < \alpha$. Of course any instance ϕ of MAX-3SAT(8) for which $n \leq n(\epsilon)$ has a constant number of variables and a constant number of clauses and therefore the exact value of $SAT(\phi)$ for such instances can be determined in $O(1)$ time. A fundamental consequence of the PCP theorem [1,2] is that there exists an α, $0 < \alpha < 1$ such that it is not possible to distinguish instances ϕ of MAX-3SAT(8) for which SAT$(\phi) = 1$ from instances for which SAT$(\phi) < \alpha$, unless P = NP. As a consequence, we have the following theorem.

Theorem 4 *There exists a $\delta > 0$ such that MDS does not have a $(1+\delta)$-approximation algorithm, unless P = NP.*

Note that in the above theorem, any value of δ, $0 < \delta < \beta$ where $\beta = \frac{(16+(1-\alpha)/24)}{16} = 1 + (1-\alpha)/384$ suffices.

3 Final Remarks

Let V' be a dominating set of a graph G. If the subgraph $G[V']$ of G induced by V', is connected, then V' is called a *connected dominating set*; if $G[V']$ has no isolated nodes, then V' is called a *total dominating set*. Keil [4] showed that minimum cardinality connected dominating set (MCDS) and minimum cardinality total dominating set (MTDS) are also NP-complete for circle graphs. The reduction described in this paper can be modified to show that MCDS and MTDS are also APX-hard. We have shown in [3] that MCDS and MTDS on circle graphs have constant-factor approximation algorithms also.

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