CS:5620 Midterm Retake Exam, Fall 2016

Monday, Dec 5, 2016

1. (a) You are given an 8-node cycle with nodes $v_1, v_2, \ldots, v_8$. For $1 < i < 8$, node v_i is connected to nodes v_{i-1} and v_{i+1}. Node v_1 is connected to nodes v_8 and v_2 and node v_8 is connected to nodes v_7 and v_1. The IDs of these nodes are as follows: $\texttt{ID}_{v_1} = 1$, $\texttt{ID}_{v_2} = 6$, $\texttt{ID}_{v_3} = 7$, $\texttt{ID}_{v_4} = 2$, $\texttt{ID}_{v_5} = 3$, $\texttt{ID}_{v_6} = 8$, $\texttt{ID}_{v_7} = 4$, and $\texttt{ID}_{v_8} = 5$. Consider the execution of the deterministic Leader Election algorithm that runs in $O(n)$ rounds and uses $O(n \log n)$ messages, on this input cycle. Recall that this algorithm runs in phases and in Phase i, every active node sends two messages, one in the clockwise direction and the other in the counter-clockwise direction, and these messages are expected to travel for 2^i hops and return. The goal of this algorithm is to elect the node with lowest ID as the leader.

   Show which nodes remain *active* after each of Phases 0, 1, and 2 of this algorithm. Your answer should consist of 3 sets of nodes: those which are active after Phase 0, those that are active after Phase 1, and those that are active after Phase 2.

   (b) Describe a family of graphs that contains for each integer $n > 1$, an edge-weighted graph G_n with n nodes for which the GHS algorithm *requires* $\Theta(\log n)$ phases.

   Recall that in a phase in the GHS algorithm, each MST fragment picks a MWOE and then the MWOEs are used to merge the MST fragments to give us new, larger MST fragments. This goes on until we have one MST fragment, which is of course an MST of the graph. This algorithm requires at most $\lceil \log_2 n \rceil$ phases, though for some inputs the algorithm could terminate much sooner, needing far fewer phases. So your task in this problem is to construct a family of examples for which the algorithm *requires* the worst-case number of phases to terminate. Don't forget to specify edge-weights in your answer.

2. (a) You are given an 10-node clique. What is the probability that Luby's MIS algorithm completes its job of finding an MIS on this graph in just one iteration? Here I am referring to the version of Luby's algorithm in which we use the notion of nodes superseding each other to break ties. To receive partial credit you must show your work.

   (b) In class, we studied the following version of Luby's $(\Delta + 1)$-coloring algorithm. Here is a informal description of a typical iteration of this algorithm.

   (i) Each as-yet-uncolored node picks a color tentatively from its current color palette, uniformly at random, and informs all neighbors.

   (ii) If a node chooses a color different from the colors chosen by all its as-yet-uncolored neighbors, then the node makes its color permanent, and informs all neighbors.

   (iii) Each node that is as-yet-uncolored removes from its palette all colors permanently chosen by neighbors.

   In class, we analyzed this algorithm to show that every as-yet-uncolored node has probability at least 1/4 of being colored in a round.

   Now consider a modification of this algorithm. At the start of an iteration, each as-yet-uncolored node goes to *sleep* with probability 1/2 and does not participate any further in the iteration. In other words, only as-yet-uncolored nodes that are awake execute the 3 steps mentioned above. Note that a node that decides to go to sleep does so for an iteration and then starts the next iteration flipping an unbiased coin to decide if it is going to sleep again.

Now consider a graph with 4 nodes in which a node v has three neighbors u_1, u_2, and u_3 and each of the nodes u_1, u_2, and u_3 have exactly one neighbor, namely v. Calculate the probability that v will be permanently colored in the first iteration of this modified Luby's $(\Delta + 1)$-coloring algorithm. Recall that a node with degree d starts off the algorithm with the palette $\{1, 2, \ldots, d+1\}$ of colors.

3. (a) Here is a claim about the GKP algorithm for constructing an MST:

    Claim: Immediately after Phases 0 and 1 of the GKP algorithm there can be an MST fragment has diameter 10.

    State whether you think this claim is True or False. Justify your answer with a brief explanation.

   (b) Here is a claim about Linial's lower bound proof on 3-coloring oriented cycles.

    Claim: The version of Linial's lower bound proof we studied in class also shows that 3-coloring an *oriented path* requires at least $\Omega(\log^* n)$ rounds.

    State whether you think this claim is True or False. Justify your answer with a brief explanation.

4. Describe a *deterministic* $O(\log n)$-round algorithm for computing a maximal matching on a planar graph in the CONGEST model. Your description can be in plain English. Feel free to use as subroutines, algorithms discussed in class or in homeworks. After your algorithm description provide a brief explanation for why your algorithm runs deterministically in $O(\log n)$ rounds.
