CS:5620 Homework 3 Solution, Fall 2016

4(a) Starting in round 1, wireless nodes will transmit in odd rounds and the base station will transmit in even rounds. Each wireless node v has a local variable $n(v)$ that is initialized to n. All n wireless nodes are initially *active*.

In an even round i, all wireless nodes are silent and the base station will transmit one of two possible messages: `failure`, if no wireless node transmitted or if more than one wireless node transmitted in round $i-1$, and `success`, if exactly one wireless node transmitted in round $i-1$. In an odd round i, each active wireless node v will first decrement $n(v)$ if in round $i-1$ the base station transmitted `success`. Furthermore, if in round $i-1$, the base station transmitted `success`, then there is exactly one wireless node that transmitted in round $i-2$ and that wireless node becomes *inactive*. Then each active wireless node v will transmit its message with probability $1/n(v)$ or stay silent with probability $1 - 1/n(v)$.

4(b) First note that the values of local variables $n(v)$ for all active nodes v are identical and equal to the number of wireless nodes currently active.

Consider the situation just before an odd round i and let $2 \leq k \leq n$ be the number of active nodes. Fix an active node v. The probability that v successfully transmits to the base station in round i is

$$\frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^{k-1} \geq \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^{k} \geq \frac{1}{4k}.$$

The probability that *some* active node successfully transmits to the base station in round i is

$$\sum_{v \text{ is active}} Pr(v \text{ successfully transmits in round } i) \geq k \cdot \frac{1}{4k} = \frac{1}{4}.$$

Now let T be a random variable that denotes the number of rounds, starting in odd round i, that it takes for one node to successfully transmit to the base station. Thus, $Pr(T = 1) \geq 1/4$. Now let X be the geometric random variable defined by

$$Pr(X = t) = \left(\frac{3}{4}\right)^{t-1} \cdot \frac{1}{4}$$

for $t = 1, 2, \ldots$. Note that $E[X] = 4$ and furthermore $E[T] \leq E[X]$. Thus, we see that in expectation, in 4 odd rounds one node will successfully transmit to the base station. By using, linearity of expectation, we see that in expected $4n$ odd rounds, all nodes will transmit to the base station. Thus the algorithm terminates in expected $8n$ rounds.