Lecture Notes CS:5350 Design and Analysis of Algorithms
Lecture 9: September 24, 2019
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1 Overview

In the previous lecture, we have talked about LP duality and the primal LP for maxflow and its dual LP. In this lecture, We'll talk about the interpretation of the dual LP as a mincut relaxation.

2 LP Formulation Based on Flow Decomposition Theorem

To interpret the dual LP in an easier way we presented in the last class an alternate LP formulation based on the flow decomposition theorem.

MaxFlow/Primal LP (A):

$$\text{maximize} \quad \sum_{P\,:\,P \text{ is an } s \rightsquigarrow t \text{ path}} x_P$$

$$\text{subject to} \quad \sum_{P\,:\,P \text{ contains } e} x_P \leq c_e, \text{ for each edge } e \in E$$

$$x_P \geq 0, \text{ for each } s \rightsquigarrow t \text{ path } P$$

Dual LP (B):

$$\text{minimize} \quad \sum_{e \in E} c_e . y_e$$

$$\text{subject to} \quad \sum_{e \in P} y_e \geq 1, \text{ for each } s \rightsquigarrow t \text{ path } P$$

$$y_e \geq 0$$

It may be worth mentioning here that (A) has exponentially many variables (in size of input graph) and (B) has exponentially many constraints.

Observation: OPT(A) = OPT(B), by strong LP duality

Let us denote the optimal value of primal LP (A) by $OPT(A)$ and the optimal value of dual LP (B) by $OPT(B)$. The observation says you can find optimal solution in both cases and therefore the values are identical.

3 Connection Between LP Duality and Maxflow-Mincut Theorem

We are trying to make the connection between LP duality and maxflow-mincut theorem. We will see that the maxflow-mincut theorem is basically a special case of more general notion of duality.

To understand the interpretation of the LP duality as a cut problem let us consider a simple example which is shown in Figure 1.

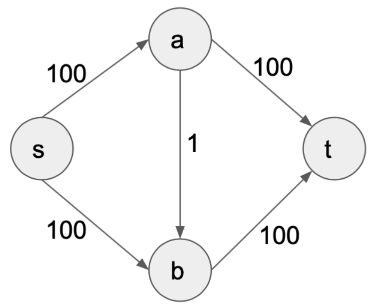

Figure 1: Flow graph with capacities c_e

Based on the Figure 1, the constraints of (B) are:

$$y_{s\to a} + y_{a\to t} \geq 1$$
$$y_{s\to a} + y_{a\to b} + y_{b\to t} \geq 1$$
$$y_{s\to b} + y_{b\to t} \geq 1$$

To get a feasible solution we can consider $y_{s\to a} = 1/2, y_{a\to b} = 0, y_{b\to t} = 1/2, y_{s\to b} = 1/2, y_{a\to t} = 1/2$ which is shown in Figure 2.

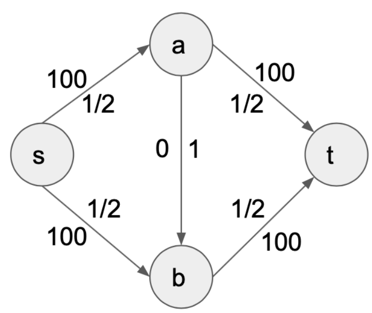

Figure 2: Graph with c_e and y_e

The cost of this solution is:

$$c_{s\to a} \cdot y_{s\to a} + c_{s\to b} \cdot y_{s\to b} + c_{a\to b} \cdot y_{a\to b} + c_{a\to t} \cdot y_{a\to t} + c_{b\to t} \cdot y_{b\to t}$$

$$= 100 \times 1/2 + 100 \times 1/2 + 0 \times 1 + 100 \times 1/2 + 100 \times 1/2 = 200$$

Note that since the maxflow in this network has value 200, this is optimal by observation. (i.e., strong LP duality)

To get an integral solution of (B), we can consider $y_{s\to a} = 1, y_{a\to b} = 0, y_{b\to t} = 1, y_{s\to b} = 0, y_{a\to t} = 0$, which is shown in Figure 3.

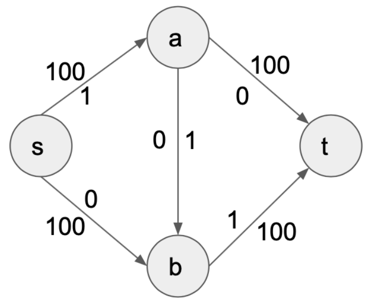

Figure 3: Graph with c_e and integral y_e

The cost of this solution is:

$$100 \times 1 + 100 \times 0 + 1 \times 0 + 100 \times 0 + 100 \times 1 = 200$$

This solution is optimal as well.

We can interpret this integral solution as defining an $(s-t)$ cut. We can interpret $y_e = 1$ as saying e is being selected for removal so as to separate s from t. We have selected edges $s \to a$ and $b \to t$ so as to cut all possible paths from s to t, as shown in Figure 4.

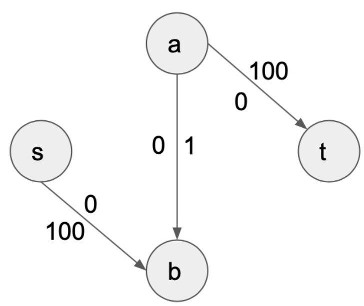

Figure 4: Graph with c_e and y_e, showing only the edges e with $y_e = 0$

From a feasible assignment $y_e \in \{0,1\}$ we can define a cut (S,T) as follows:

- $S = \{v \in V |$ there is a path from s to v only containing edges e with $y_e = 0\}$
- $T = V \setminus S$

In the above example, $S = \{s, b\}$ and $T = \{a, t\}$. Note that in general since $\sum_{e \in P} y_e \geq 1$ for every $s \rightsquigarrow t$ path P, S does not contain t. So, we can interpret the solution of (B) as a cut if we place the additional restriction that $y_e \in \{0,1\}$ for all e.

3.1 Formalizing the Connection Between Integer Program and Mincut Problem

To make some of these ideas more precise let us define an integer program by replacing the non-negativity constraints in (B) with integrality constraints $y_e \in \{0,1\}$. Let us call this integer program: (C) and argue that (C) models the mincut problem. An integer program (IP) refers to a mathematical program with linear objective function, linear constraints, and *integrality constraints*. We formalize the connection between integer program (C) and the mincut problem as follows:

Lemma 1 *Let (S^*, T^*) be a cut with minimum capacity then $OPT(C) = ||S^*, T^*||$.*

Proof: (a) First, let us prove $OPT(C) \leq ||S^*, T^*||$. From the cut (S^*, T^*), let us define the following assignment of values to y_e which is shown graphically in Figure 5:

- $y_e = 0$, if both endpoints of e are in S^* or both endpoints of e are in T^* or e goes from T^* to S^*
- otherwise, y_e=1

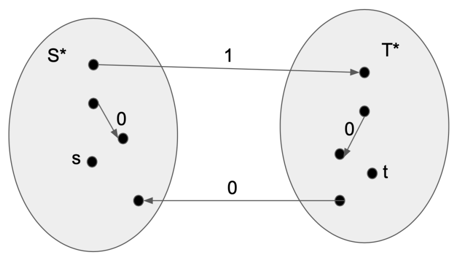

Figure 5: Assignment of values to y_e

It is easy to check that this solution satisfies the constraints and is therefore a feasible solution to (C). It is also easy to check that the objective function value for this feasible solution is $\sum\limits_{e \in E \cap (S^* \times T^*)} c_e = ||S^*, T^*||$. Therefore, by definition $OPT(C) \leq ||S^*, T^*||$.

(b) Now, let us prove $OPT(C) \geq ||S^*, T^*||$. Consider an optimal solution $y_e^* \in \{0, 1\}$ of (C). Let us define S and T:

- $S = \{v | v \text{ is reachable from } s \text{ only using edges } e : y_e^* = 0\}$
- $T = V \setminus S$

It is easy to check that (S, T) is an (s, t)-cut, i.e., a cut that separates s from t. Since (S^*, T^*) by definition is a mincut so $||S^*, T^*|| \leq ||S, T||$. We also know that for every edge e going from S to T, $y_e^* = 1$; otherwise the endpoint of e in T would have been in S. Therefore, we know that $||S, T|| \leq \sum\limits_{e \in E} c_e . y_e^* = OPT(C)$. Therefore $||S^*, T^*|| \leq OPT(C)$. So, we can conclude $OPT(C) \geq ||S^*, T^*||$. $\square$

3.2 OPT(C)=OPT(B)

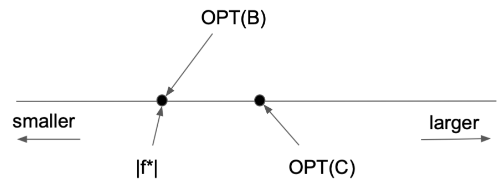

Figure 6: Graphical representation of $OPT(C) \geq OPT(B)$

Figure 6 represents graphically, what we have really discussed till now. Here, $|f^*|$ is the maxflow value and there is an LP corresponds to this. Then, we wrote the dual (B) of this LP. By strong duality, we get $|f^*| = OPT(B)$. After that, we wrote integer program (C) of this LP. Since, feasible solutions of (B) is a superset of feasible solutions of (C) and (B) and (C) are minimization problems, we can say $OPT(C) \geq OPT(B)$. We already know the maxflow-mincut theorem which shows that $|f^*| = ||S^*, T^*||$ and we know that $OPT(C) = ||S^*, T^*||$ by earlier lemma. This implies that $OPT(B) = OPT(C)$ as shown in Figure 7.

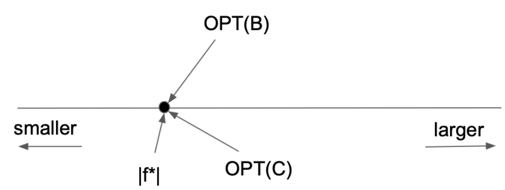

Figure 7: Graphical representation of $OPT(C) = OPT(B)$

In otherwords, we have shown that the LP (B) has an integral optimal solution. We state this as a theorem.

Theorem 2 *The LP (B) has an integral solution , i.e., a solution* $y_e \in \{0, 1\}$, *that is optimal.*