Lecture Notes CS:5350 Minimum Vertex Cover: 2-approximation and LP-based views
Lecture 20: October 31, 2019
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1 Minimum Vertex Cover (MVC)

Input: A graph $G = (V, E)$
Output: A vertex subset S of smallest size such that for all $\{u, v\} \in E$, at least one of u or v is in S.

Minimum Vertex Cover is one of 21 problems shown to be NP-Complete by Karp in 1972.

A simple algorithm for approximating MVC is as follows,

(i) Find a maximal matching M for the input graph G. Recall that set M is the *maximal* subset of edges such that no two edges in M do share a common vertex.

(ii) For every edge $\{u, v\} \in M$, add both u and v to vertex cover S.

(iii) Output the computed vertex cover S.

It is easy to see that this algorithm runs in $\mathcal{O}(m + n)$ time.

Theorem. *The algorithm described above is a 2-approximation algorithm for MVC.*

Example: Consider the graph shown in Figure 1.

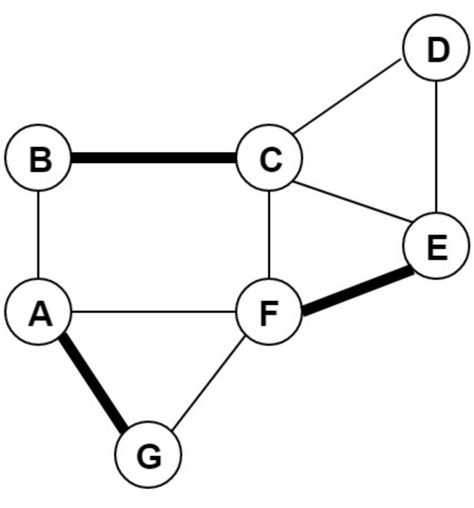

Figure 1: Example graph G, and a maximal matching M, where bold edges belong to M

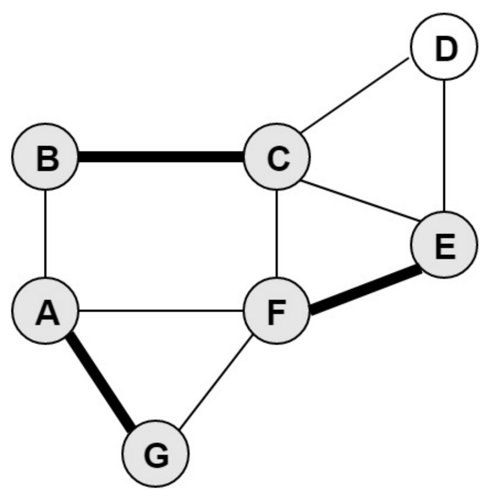

Figure 2: Highlighted vertices belonging to the computed vertex cover S

Figure 1 shows edges (in bold) of M, which is some maximal matching computed by our algorithm in step 1. Based on this matching, the algorithm outputs the set $S = \{B, C, E, F, A, G\}$. This is shown in Figure 2.

Proof. Given an optimal MVC S^* for G, we prove that our algorithm produces a vertex cover S such that $|S| \leq 2.|S^*|$. We do this by first stating and proving two lemmas related to our algorithm, and the general relationship between matchings and vertex covers.

Lemma 1. *S is a vertex cover*

Proof. We prove this by contradiction. Suppose some edge $e = \{u, v\} \in E$ is not covered by S, i.e., both $u \notin S$ and $v \notin S$. As both u and v are not in S, any edges incident on u and v, including e, are not part of our matching M, which means that we can add $e = \{u, v\}$ to M without violating the matching property. However, in doing this, we are violating the maximality of the matching, which means that no such edge e exists, hence S is indeed a vertex cover. □

Lemma 2. *For any matching M and any vertex cover S of graph G, $|M| \leq |S|$.*

Proof. Since M by definition is set of disjoint edges, i.e., no two edges share a common vertex, and a vertex cover S touches all edges at least once, for each edge $e = \{u, v\} \in M$, at least one of u and v has to be included in S. Therefore, size of S is at least as much as the size of M. □

From Lemma 2, for a graph G we can also say that any arbitrary matching M computed by our algorithm, and S^* a Minimum Vertex Cover of G satisfies the property

$$|M| \leq |S^*| \tag{1}$$

The following results from how our algorithm works and equation 1,

$$|S| = 2|M| \leq 2|S^*|$$

This means that our algorithm produces a 2-approximation of MVC. □

For decades, no improvement has been made to improve the factor of 2 approximation for MVC. This has indirectly led the research community to conjecture that no improvement beyond 2 is indeed possible. This conjecture which is related to the Unique Games Problem, in one form states that if the Unique Games Problems is shown to be NP-complete, then we cannot get a better approximation factor for MVC than 2.

2 Linear Programming view of MVC

An Integer Program for MVC (MVC-IP)

With choice variables as $x_v \in \{0, 1\} \forall v \in V$, the objective is to minimize the cost function subject to conditions as follows,

$$\begin{aligned} &\text{cost function:} && \text{minimize} \sum_{v \in V} x_v && \\ &\text{subject to} && x_u + x_v \geq 1 && \text{for each edge } \{u, v\} \in E \\ &\text{and} && x_v \in \{0, 1\} && \text{for each } v \in V \end{aligned}$$

LP-relaxation of MVC-IP (MVC-LP)

Relaxing the integrality constraint on choice variables in MVC-IP to non-zero constraints,

$$\begin{array}{rcr} \text{cost function:} & \text{minimize} \displaystyle\sum_{v \in V} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each edge } \{u, v\} \in E \\ \text{and} & x_v \geq 0 & \text{for each } v \in V \end{array}$$

Dual of MVC-LP

With choice variables as y_e for each $e \in E$ we get the following cost function subject to constraints shown below:

$$\begin{array}{rcr} \text{cost function:} & \text{maximize} \displaystyle\sum_{e \in E} y_e & \\ \text{subject to} & \displaystyle\sum_{e:\ e \text{ incident on } v} y_e \leq 1 & \text{for each } v \in V \\ \text{and} & y_e \geq 0 & \text{for each } e \in E \end{array}$$

Now, for some graph $G = (V, E)$, any matching M is a feasible solution for the Dual MVC-LP, and any vertex cover S is a feasible solution to the MVC-LP.

Therefore, by LP weak duality theorem, we obtain the same result as Lemma 2, as follows:

$$|M| \leq |S|$$

2.1 Primal-dual method for finding a vertex cover

Simply stated, the algorithm is as follows,

(1) Find a feasible solution for Dual MVC-LP.

(2) Extract from this a feasible integral primal solution, i.e., a solution for MVC-IP, not too much larger in cost than the cost of the solution in (1). (This is motivated by the fact that the cost of any feasible dual solution is a lower bound on the optimal integral primal solution.)

A dual of MVC-IP with matching set M would produce a vertex cover S such that $|S| = 2|M|$ (which is the 2-approximation theorem for MVC). Visually, this looks like Figure 3 shown below.

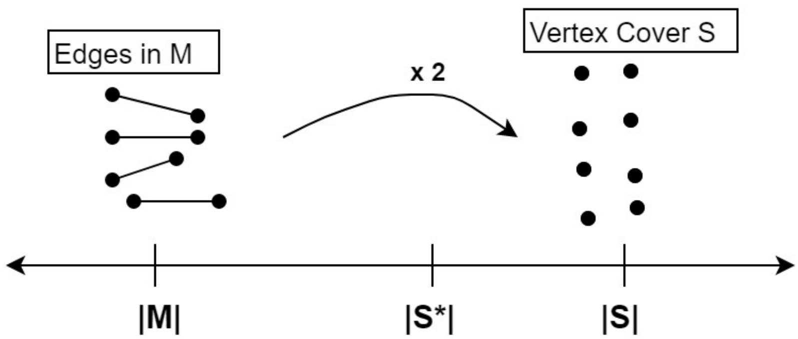

Figure 3: A cost-line view of sets M and S obtained from the primal-dual method

2.2 LP-rounding approximation for MVC

This algorithm is as follows,

(i) Solve MVC-LP to obtain solution $\{x_v | v \in V\}$

(ii) for each $v \in V$: if $x_v \geq 1/2$, set $z_v \leftarrow 1$
if $x_v \leq 1/2$, set $z_v \leftarrow 0$

(iii) Output $S = \{v | z_v = 1\}$

Lemma 3. *The set S output by the LP-rounding alorithm is a vertex cover.*

Proof. Feasibility of the MVC-LP solution tells us that for every edge $\{u, v\} \in E$, either $x_u \geq 1/2$ or $x_v \geq 1/2$. Therefore, for every edge $\{u, v\} \in E$ either $z_u = 1$ or $z_v = 1$. In other words either $u \in S$ or $v \in S$. □

Lemma 4. *Let S be the output of LP-rounding, and S^* be a minimum vertex cover. Then,*

$$|S| \leq 2|S^*|.$$

Proof. Note that, $\sum_{v \in V} x_v \leq |S^*|$

since $\{x_v | v \in V\}$ is an optimal solution to MVC-LP, which is a relaxation of the integer program MVC-IP.

Also, $z_v \leq 2x_v$. Therefore,

$$\sum_{v \in V} z_v \leq 2 \sum_{v \in V} x_v \leq 2|S^*|$$

Hence,
$$|S| \leq |S^*|.$$

□