CS:5350 Homework 1, Fall 2019

Due in class on Tue, Sep 17

Notes: (a) It is possible that solutions to some of these problems are available to you via textbooks, on-line lecture notes, etc. If you use any such sources even partially, please acknowledge these in your homework fully *and* present your solutions in your own words. You will benefit most from the homework, if you seriously attempt each problem on your own first, before seeking other sources. (b) It is okay to form groups of **three** in solving and submitting homework solutions. (The syllabus says groups of at most two are allowed, but given the size of the class and the fact that the class has no TA, I am modifying this requirement to allow groups of three.) But, my advice from (a) still applies: you will benefit most from the homework, if you seriously attempt each problem on your own first, before seeking help from your group partner. (c) Unless you have a documented accomodation, no late submissions are permitted. You will receive no points for your submission if your submission is not turned in at the beginning of class on the due date. (d) Your submissions will be evaluated on correctness *and* clarity. Correctness is of course crucial, but how clearly you communicate your ideas is also quite important.

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1. Given two feasible (s, t)-flows f_1 and f_2 in a graph $G = (V, E)$, define the *average flow* $\bar{f}$ as
$$\bar{f}(u \to v) = \frac{1}{2} \cdot f_1(u \to v) + \frac{1}{2} \cdot f_2(u \to v)$$
for every edge $u \to v \in E$. Is $\bar{f}$ a feasible flow? Support your answer either with a proof or with a counterexample.

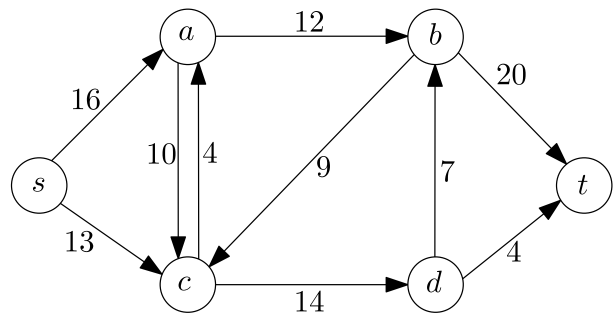
2. Consider the following *two-source maximum flow* problem, which we will call 2SourceMaxFlow. You are given a directed graph $G = (V, E)$ and three special vertices $s_1, s_2, t \in V$. Think of s_1 and s_2 as two sources and t and the single target or sink. A flow is defined as before, except that flow conservation need not hold at s_1, s_2, or t. The definition of a *feasible* flow is as before. The *value* of the flow is the net outflow at s_1 plus the net outflow at s_2. As in the case of the MaxFlow problem, in order to solve the 2SourceMaxFlow problem we want to find a feasible flow with maximum value.

   Describe a polynomial-time algorithm for this problem, by efficiently *reducing* it to MaxFlow. In other words, given an input $\langle G = (V, E), s_1, s_2, t, c : E \to \mathbf{R}_{\geq} \rangle$ to 2SourceMaxFlow, (a) show that it can be modified in polynomial time into an input $\langle G' = (V', E'), s', t', c' : E' \to \mathbf{R}_{\geq} \rangle$ to MaxFlow and (b) from the solution to MaxFlow on this input, we can extract, again in polynomial-time a solution to 2SourceMaxFlow on the original input.

3. The input is a directed graph $G = (V, E)$, special vertices $s, t \in V$, and *two* capacity functions $c_1, c_2 : E \to \mathbf{R}_{\geq}$. A flow $f : E \to \mathbf{R}$ is said to be *feasible* for the input if $c_1(u \to v) \leq f(u \to v) \leq c_2(u \to v)$ for all edges $u \to v \in E$. (Recall that a flow, by definition, satisfies flow conservation at all vertices except s and t.)

   Show that there is a polynomial-time algorithm for determining if there is a feasible flow for the input.

4. Consider the directed graph G given below, with source s and target t identified.



(a) Let f be a flow in G defined by $f(s \to a) = 12$, $f(a \to b) = 12$, $f(b \to t) = 12$ and $f(u \to v) = 0$ for all other edges $u \to v$ in G. Draw the residual graph G_f.

(b) Identify an augmenting path P in G_f that has the fewest number of edges. Define a new flow f' obtained by augmenting f along path P. Draw the residual graph $G_{f'}$.

(c) Is f' a maximum flow? If your answer is "yes," prove it by showing a cut with capacity equal to the value of f'. If your answer is "no," then identify a flow f'' such that $|f''| > |f'|$.

5. Problem 17 from Chapter 10 in Prof. Erickson's notes.

6. Problem 16(a) from Chapter 11 in Prof. Erickson's notes.
Hints: (i) Think about how we reduced the Baseball Elimination problem to a flow problem. (ii) Even if you cannot reduce this matrix rounding problem the standard MaxFlow problem, consider reducing to one of the MaxFlow variants described earlier in this handout.