22C:296 Lecture Notes 09/17

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- Proof of Lovasz local lemma (L^3).
- Proof of symmetric L^3.
- Application: vertex-disjoint cycles

L^3: let $A_1, A_2, ..., A_n$ be events in same probability space. Let $D = (V, E)$ be a directed graph for $\{A_1, A_2, ..., A_n\}$, suppose there are real numbers $x_1, x_2, ..., x_n$ $0 \leq x_i < 1$ and $Prob[A_i] \leq x_i \Pi_{(i,j)\in E}(1 - x_j)$, then $Prob[\wedge_{i=1}^n \bar{A}_i] \geq \Pi_{i=1}^n (1 - x_i) > 0$

Proof: By induction on s, we can prove that $Prob[A_i | \wedge_{j\in S} \bar{A}_j] \leq x_i$ for all i, $S \subseteq \{1, 2, ..., n\}$, $|S| = s$, $i \notin S$.

From this L^3 follows because:

$$\begin{aligned} Prob[\wedge_{i=1}^n \bar{A}_i] &= Prob[\bar{A}_1 | \bar{A}_2 \wedge \bar{A}_3 \cdots \wedge \bar{A}_n] \cdot Prob[\bar{A}_2 \wedge \bar{A}_3 \cdots \wedge \bar{A}n] \\ &= Prob[\bar{A}_1 | \wedge_{j=2}^n \bar{A}_j] \cdot Prob[\bar{A}_2 | \wedge_{j=3}^n \bar{A}_j] \cdots Prob[\bar{A}_n] \\ &\geq (1 - x_1)(1 - x_2) \cdots (1 - x_n) \\ &> 0 \end{aligned}$$

For $s = 0$, the claim is trivially true. We assume that the claim is true for all $s' < s$ and show it is true for s.

Let S be partitioned into subsets $S_1 = \{j \in S | (i, j) \in E\}$ and $S_2 = S - S_1$

$$\begin{aligned} Prob[A_i | \wedge_{j\in S} \bar{A}_j] &= \frac{Prob[A_i \wedge (\wedge_{j\in S} \bar{A}_j)]}{Prob[\wedge_{j\in S} \bar{A}_j]} \\ &= \frac{Prob[A_i \wedge (\wedge_{j\in S_1}) \bar{A}_j | (\wedge_{j\in S_2})] \cdot Prob[\wedge_{j\in S_2} \bar{A}_j]}{Prob[\wedge_{j\in S_1} \bar{A}_j | \wedge_{j\in S_2}] \cdot Prob[\wedge_{j\in S_2} \bar{A}_j]} \\ &= \frac{Prob[A_i \wedge (\wedge_{j\in S_1}) \bar{A}_j | (\wedge_{j\in S_2})]}{Prob[\wedge_{j\in S_1} \bar{A}_j | \wedge_{j\in S_2}]} \end{aligned}$$

For the numerator,

$$\begin{aligned} Prob[A_i \wedge (\wedge_{j\in S_1} \bar{A}_j) | (\wedge_{j\in S_2} \bar{A}_j)] &\leq Prob[A_i | (\wedge_{j\in S_2} \bar{A}_j)] \\ &= Prob[A_i] \\ &\leq X_i \Pi_{(i,j)\in E}(1 - x_j) \end{aligned}$$

For the denominator, if $S = \emptyset$, the probability is 1, and we are done. So we assume that $|S| \geq 1$. Let $S_1 = \{A_{j_1}, A_{j_2}, ..., A_{j_r}\}$

$$\begin{aligned}
Prob[\wedge_{j\in S_1}\bar{A}_j|\wedge_{j\in S_2}] &= Prob[A_{j_r}|\wedge_{j\in S_2}\bar{A}_j]\cdot Prob[A_{j_{r-1}}^{-}|\bar{A}_{j_r}\wedge(\wedge_{j\in S_2}\bar{A}_j)] \\
&\quad . \\
&\quad . \\
&\quad . \\
&\quad \cdot Prob[A_{j_1}|(\bar{A}_{j_2}\wedge\bar{A}_{j_2}\wedge\cdots\wedge\bar{A}_{j_r})\wedge(\wedge_{j\in S_2}\bar{A}_j)]] \\
&\geq (1-x_{j_r})(1-x_{j_{r-1}})\cdots(1-x_{j_1}) \\
&\geq \Pi_{(i,j)\in E}(1-x_j)
\end{aligned}$$

□

L^3 symmetric version

Let $A_1, A_2, ..., A_n$ be events in a probability space such that each A_i is independent of all but most d other events. If $Prob[A_i] \leq p$ for each i, and $ep(d+1) \leq 1$, then $Prob[\wedge_{j=1}^n \bar{A}_j] > 0$

Proof: This can be derived from the asymmetric version as follows:

If $d = 0$, then the result is obvious.

If $d \geq 1$, then set $x_i = \frac{1}{d+1} < 1$. We will verify $Prob[A_i] \leq X_i\Pi_{(i,j)\in E}(1-x_j)$

$$\begin{aligned}
R.H.S &= X_i\Pi_{(i,j)\in E}(1-x_j) \\
&= \frac{1}{d+1}\Pi_{(i,j)\in E}(1-\frac{1}{d+1}) \\
&\geq \frac{1}{d+1}(1-\frac{1}{d+1})^d \\
&\geq \frac{1}{d+1}\cdot\frac{1}{e} \\
&= \frac{1}{e(d+1)}
\end{aligned}$$

Given that $ep(d+1) \leq 1$, so $p \leq \frac{1}{e(d+1)}$, thus original $R.H.S \geq p \geq Prob[A_i]$. Hence asymmetric L^3 holds and $Prob[\wedge_{j=1}^n \bar{A}_j]$ □

Application

Theorem 1 *Every k-regular directed graph G has a collection of $\lfloor \frac{k}{3\ln k} \rfloor$ vertex-disjoint cycles.*

Definition 2 *A k-regular directed graph is a digraph in which $out-degree(v) = in-degree(v) = k$ for all vertices $v \in V$.*

Proof: Let $C = \lfloor \frac{k}{3\ln k} \rfloor$, construct a partition of V into subsets $V_1, V_2, ..., V_C$ randomly, by picking each vertex and throwing it uniformly at random into one of $V_1, V_2, ..., V_C$.

We will show with positive probability, each $G[v_i]$ has a cycle. To show this, we will show that with > 0 probability, each vertex has an outneighbor in the same part.

Let $A_v \equiv v$ does not have an outneighbor in the same part.
Showing that $Prob[\wedge_{v \in V} \bar{A}_v] > 0$ suffices to show our claim

$$Prob[A_v] = (1 - \frac{1}{C})^k \leq e^{-\frac{1}{C} \cdot k} = e^{-\frac{k}{\frac{k}{3 \ln k}}} = k^{-3}$$

A_v is mutually independent of all events in

$$\{A_u | \overline{(u \cup N^+(u))} \cap (v \cup N^+(v)) = \emptyset\}$$

So total number of events not independent with A_v is $(k+1) \cdot k + (k+1) = (k+1)^2$. Which means A_v is mutually independent of all but most $(k+1)^2$ events. We now only need to verify that

$$e \cdot \frac{1}{k^3} \cdot ((k+1)^2 + 1) \leq 1$$

which is true for $k \geq 6$.

□