Lazy Select

- Pick $n^{3/4}$ elements from S independently and uniformly at random with replacement into R.
- Sort R.
- Let $x = kn^{-1/4}$, $l = \max\{\llcorner x - \sqrt{n}\lrcorner, 1\}$, $h = \min\{\ulcorner x + \sqrt{n}\urcorner, n^{3/4}\}$, $a = R_l$ and $b = R_h$. By comparing every element in S with a determine $r_s(a)$. Similarly determine $r_s(b)$.
- If $k < n^{1/4}$, then $P = \{y \in S \mid y \leq b\}$.
  If $k > n - n^{1/4}$, then $P = \{y \in S \mid y \geq a\}$.
  If $k \in [n^{1/4}, n - n^{1/4}]$, then $P = \{y \in S \mid a \leq y \leq b\}$
  Check if $S_k \in P$ and $\mid p \mid \leq 4n^{3/4} + 2$ otherwise repeat [1] to [3].
- Sort P and return $P_{(k - r_s(a)+1)}$.

Running time There are $2n$ comparisions in step (3) $+o(n)$.

Since the condition in step (3) may fail with > 0 probability, we want to computer the expected number of comparisions.

$$(2n + o(n)) + O(n^{-1/4})(2n + o(n)) + ... = 2n + o(n)$$

Claim: $P[(S_k \in P) \wedge (\mid P \mid \leq (4n^{3/4} + 2)] \geq 1 - O(n^{-1/4})$

Proof: we will show

$$P[(S_k \notin P) \vee (\mid P \mid > 4n^{3/4} + 2)] \leq O(n^{-1/4})$$

To show this we will show

$$P[S_k \notin P] \leq O(n^{-1/4})]$$

$$P[\mid P \mid > 4n^{3/4} + 2] \leq O(n^{-1/4})]$$

For $P[S_k \notin P] \leq O(n^{-1/4})]$

There are 3 cases depending on whether $k < n^{1/4}$, $k > n - n^{1/4}$, $k \in [n^{1/4}, n - n^{1/4}]$. The last case is hardest and we will show only this case.

For the last case, given $k \in [n^{1/4}, n - n^{1/4}]$ and therefore $P = \{y \in S \mid a \leq y \leq b\}$. $S_k \notin P \equiv \neg(a \leq S_k \leq b) \equiv (S_k < a) \vee (S_k > b)$. We will now show $P[S_k < a] \leq O(n^{-1/4})$. Similarily, we can show $P[S_k > b] \leq O(n^{-1/4})$.

$$S_k < a \equiv S_k < R_l \equiv \text{there are less than } l \text{ elements in } R \leq S_k$$

Let

$$x = \begin{cases} 1 & \text{if } i^{th} \text{ sample} \leq S_k \\ 0 & \text{otherwise} \end{cases}$$

Then $P[x_i = 1] = \frac{k}{n}$, $E[x_i] = \frac{k}{n}$. Let

$$X = \sum_{i1}^{n^{3/4}} x_i$$

. Then

$$E[X] = \sum E[x_i] = kn^{-1/4}$$

. So the event:

$$\text{there are fewer than } l \text{ element in } R \leq S_k$$

$$\equiv X < l \equiv X < X - \sqrt{n} \equiv X < kn^{-1/4} - \sqrt{n} \equiv (X - E[X]) < -\sqrt{n}$$

By Chebyshev Inequality:

$$P[\mid X - E[x] \mid \geq t] \leq \frac{Var[X]}{t^2}$$

Let $t = \sqrt{n}$

$$Var[X] = Var[\sum_{i1}^{n^{3/4}} x_i]$$

Because of mutual independence of x_i

$$Var[X] = \sum_{i1}^{n^{3/4}} Var[x_i] = \sum_{i1}^{n^{3/4}} (E[x_i^2] - E[x_i]^2)$$

$$Var[X] = n^{3/4}(\frac{k}{n} - \frac{k^2}{n^2}) = \frac{k}{n}(1 - \frac{k}{n})n^{3/4} \leq \frac{n^{3/4}}{4}$$

So:

$$Var[X] \leq \frac{n^{3/4}}{4}$$

$$P[\mid X - E[X] \mid \geq \sqrt{n}] \leq \frac{n^{-1/4}}{4}$$

So:

$$P[X - E[X] < -\sqrt{n}] \leq \frac{n^{-1/4}}{4}$$

□

Lovasz Local Lemma Let $A_1, ..., A_n$ be event in a arbitrary probability space. A direct graph $D = (V, E)$, $V = \{1, 2, ..., n\}$ is said to be the dependency graph of $A_1, ..., A_n$ if for each i, A_i is mutually independent of $\{A_j \mid (i, j) \notin E\}$.

Suppose that $D = (V, E)$ is a dependency graph of the above event and suppose $\exists x_1, x_2, ..., x_n \in R$ such that $0 \leq x_i < 1$ for all i and $P[A_i] \leq x_i \prod_{(i,j) \in E}(1 - x_j)$ for all i, $1 \leq i \leq n$. Then:

$$P[\wedge_{j1}^{n} \overline{A_j}] \geq \prod_{i1}^{n}(1 - x_i) > 0$$

Event A_1 is said to be mutually independent of $\{A_2, ..., A_n\}$, if for any subset $S \subseteq \{2, ..., n\}$,

$$P[A_1 \mid \wedge_{j \in S} E_j] = P[A_1]$$

where $E_j \in \{A_j, \overline{A_j}\}$

If A_i are all mutually independent,

$$P[\wedge_{i1}^{n} \overline{A_i}] = \prod_{i1}^{n} P[\overline{A_i}] = \prod_{i1}^{n}(1 - P[A_i])$$

If $P[A_i] < 1$, then we know the above is > 0.