22C:296 Seminar on Randomization

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We continue to discuss the basic idea behind the probabilistic method. In the last class, we showed that if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then $R(k,k) > n$.

Two remarks about the proof:

1. The proof uses the *union bound*

$$\text{Prob}[\bigcup_i A_i] \leq \sum_i \text{Prob}[A_i]$$

   Here each A_i is an event and we make no assumptions about the independence of these events.

2. A simple counting argument also shows the same result. The number of 2-color edge-coloring's of K_n is $2^{\binom{n}{2}}$. The number of 2-color edge-coloring's of K_n containing a monochromtic K_k is bounded above by $\binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \cdot 2$. Therefore, if n and k are natural numbers such that If $\binom{n}{2} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \cdot 2 < 2^{\binom{n}{2}}$ then there exists a 2-color edge-coloring of K_n which does <u>not</u> contain a monochromatic K_k. After simplifying $\binom{n}{2} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \cdot 2 < 2^{\binom{n}{2}}$, we get $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$.

Corollary 1 $R(k,k) > 2^{\frac{k}{2}}$.

Proof: This is merely algebra. One approach is to simply find n (as a function of k) satisfying $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$. To do this use Stirling's approximation for $\binom{n}{k}$.

Alternately, substitute $n = \lfloor 2^{\frac{k}{2}} \rfloor$ in the above inequality, we get

$$\begin{aligned}
\binom{n}{k} \cdot 2^{1-\binom{k}{2}} &= \frac{n(n-1)\cdots(n-(k-1))}{k!} \cdot 2 \cdot 2^{-\frac{k^2}{2}} \cdot 2^{\frac{k}{2}} \\
&\leq \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \\
&\leq \frac{2^{1+\frac{k}{2}}}{k!} \\
&< 1 \; (for\ k \geq 3)
\end{aligned}$$

□

Questions: Since $R(k,k) > 2^{\frac{k}{2}}$, if we let $n = 2^{k/2}$, then there is a 2-color edge-coloring of K_n that does not contain a monochromatic K_k. In other words, there is a 2-color edge-coloring of K_n that does not contain a monochromatic $K_{2\log n}$ For example, there is a 2-color edge coloring of K_{1024} that doesn't contain a monochromatic K_{20}. The algorithmic question is how can we efficiently find such a coloring?

Example 2: Tournaments

Definition 2 *A tournament is an orientation of the edges of a complete graph. If (x, y) is an edge in the tournament, then we say that x "beat" y.*

Definition 3 *A tournament is said to have a property S_k, if for every set of k players, there is a player who has beaten them all.*

Questions:

1. For every k, is there a finite n such that there is a tournament on n vertices with property S_k?
2. If so, what is the smallest n for which this is true?

   Schülte asked this in 1963 and Erdös provided an easy answer using the probabilistic method.

Theorem 4 *If $\binom{n}{k}\cdot (1-2^{-k})^{n-k} < 1$, then there is a tournament on n vertices with property S_k.*

Proof: Suppose $\binom{n}{k}\cdot (1-2^{-k})^{n-k} < 1$, construct random orientation of K_n by independently orienting each edge in one of two directions with equal probability. We get probability space $2^{\binom{n}{2}}$ elements with each element has a probability of $\frac{1}{2^{\binom{n}{2}}}$.

Fix a set of k vertices, call this set K. Let A_k denote the event that there is no vertex that beats every vertex in K. Let v denote a vertex outside of K.

What is Prob$[A_k]$?

$$\begin{aligned}
\text{Prob}[v \textit{ doesn't beat everyone in } K] &= 1 - (\frac{1}{2})^k \\
\text{Prob}[A_k] &= \text{Prob}[\wedge_p \; v \textit{ doesn't beat everyone in } K] \\
&= \prod_{v \notin K} \text{Prob}[v \textit{ doesn't beat everyone in } K] \\
&= (1 - \frac{1}{2^k})^{n-k} (\textit{because of independent events})
\end{aligned}$$

In order to show that Prob[*for any set of k vertices, there exists a vertex beats them all*] > 0, we want to show the following complementary claim Prob[*there is a set of k vertices, such that no vertex beats everyone in the set*] < 1 is true.

$$
\begin{aligned}
\text{Prob}[\textit{there is a set of } k \textit{ vertices, such that no vertex beats them all}] &= \text{Prob}[\vee A_k] \\
&\leq \sum_{K:|K|=k} \text{Prob}[A_k] \\
&= \binom{n}{k} \cdot (1-2^{-k})^{n-k} \\
&< 1
\end{aligned}
$$

Since the complemetary claim is true, then we know Prob[*for any set of k vertices, there exists a vertex beats them all*] > 0 is true. □