22C:296 Seminar on Randomization

Scribe: Darren Schipper

October 13, 2003

We wish to show that any k-regular digraph with no parallel edges has at least $\Omega(k^2)$ edge-disjoint cycles.

Lemma 1 *Let G be a digraph with no parallel edges, min degree $\geq k \geq 1$, and max degree $\leq 2k$. Then the vertices of G may be colored with at least $\frac{k}{2^{16}}$ colors, each used in such a way that for each color, the corresponding induced subgraph has vertex degrees in the range $[a, 4a]$ where $a \geq 1$.*

Before proving this, we will see why this implies our result. It implies at least $\frac{k}{2^{16}}$ edge-disjoint cycles.
For $j = \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil + 1, \ldots k$, we define the digraph G_j as follows:
$G_k = G$
G_j where $j \in [\lceil \frac{k}{2} \rceil, k-1]$ defined recursively. Suppose G_{j+1} satifies the hypothesis that min degree of any vertex $\geq j+1$, and max degree $\leq 2(j+1)$. Using the lemma, we get $\frac{j+1}{2^{16}}$ edge-disjoint cycles. Delete these from G_{j+1} to get G_j.

Since each vertex in G_{j+1} has its vertex degree reduced by at most 1, the min vertex degree of any vertex in $G_j \geq j$, and the max vertex degree $\leq 2k$. Since $j \geq \lceil \frac{k}{2} \rceil, 2j \geq 2k$. Hence, max vertex degree of $G_j \leq 2j$. In this manner, we get

$$\sum_{j=\lceil \frac{k}{2} \rceil}^{k} \frac{j}{2^{16}} \geq \frac{3k^2}{2^{19}} \tag{1}$$

To prove Lemma 1, we first prove the following intermediate result. The result uses CH bounds and L^3.

Lemma 2 *Suppose H is a digraph with no parallel edges, min degree $\geq x$, and max degree $= y$, where $x \geq 1000$ and $y \leq 4x$. Then the vertices of H can be colored red or blue such that*

$$\begin{aligned} \forall v \in V(H): \quad & \textit{\# red out neighbors} \in [\frac{\delta^+(v)}{2} - \delta^+(v)^{\frac{2}{3}}, \frac{\delta^+(v)}{2} + \delta^+(v)^{\frac{2}{3}}] \\ \textit{and} \quad & \textit{\# red in neighbors} \in [\frac{\delta^-(v)}{2} - \delta^-(v)^{\frac{2}{3}}, \frac{\delta^-(v)}{2} + \delta^-(v)^{\frac{2}{3}}] \\ \textit{and} \quad & \textit{similarly for blue} \end{aligned}$$

Proof: For each vertex $v \in V(H)$ color v red or blue randomly, independently, and with equal probability. For each $v \in V(H)$, let X_v^+ denote the number of red out neighbors of v.

$$E[X_v^+] = \frac{\delta^+(v)}{2} \tag{2}$$

Similarly for X_v^- denoting the red in neighbors of v.

Define the following events:

$$A_v^+ : X_v^+ \ni [\frac{\delta^+(v)}{2} - \delta^+(v)^{\frac{2}{3}}, \frac{\delta^+(v)}{2} + \delta^+(v)^{\frac{2}{3}}]$$
$$A_v^- : X_v^- \ni [\frac{\delta^-(v)}{2} - \delta^-(v)^{\frac{2}{3}}, \frac{\delta^-(v)}{2} + \delta^-(v)^{\frac{2}{3}}]$$

Use L^3 to show

$$Prob[(\bigwedge_{v\in V} \overline{A_v^+}) \wedge (\bigwedge_{v\in V} \overline{A_v^-})] > 0 \tag{3}$$

We must bound probability of A_v^+ and A_v^- first.

$$\begin{aligned} Prob[A_v^+] &= Prob[(X_v^+ > \frac{\delta^+(v)}{2} + \delta^+(v)^{\frac{2}{3}}) \vee (X_v^+ < \frac{\delta^+(v)}{2} - \delta^+(v)^{\frac{2}{3}})] \\ &\leq Prob[X_v^+ > \frac{\delta^+(v)}{2} + \delta^+(v)^{\frac{2}{3}}] + Prob[X_v^+ < \frac{\delta^+(v)}{2} - \delta^+(v)^{\frac{2}{3}}] \end{aligned}$$

Each of these can be bounded using CH bounds. For example,

$$\begin{aligned} Prob[X_v^+ > \frac{\delta^+(v)}{2} + \delta^+(v)^{\frac{2}{3}}] &= Prob[X_v^+ > \frac{\delta^+(v)}{2}(1 + 2\delta^+(v)^{-\frac{1}{3}})] \\ &< \frac{e^{2\delta^+(v)^{-\frac{1}{3}}}}{(1+2\delta^+(v)^{-\frac{1}{3}})^{1+2\delta^+(v)^{-\frac{1}{3}}}} \\ &\leq e^{-2\delta^+(v)^{\frac{1}{3}}} \end{aligned}$$

We get the same upper bound for the second term to get

$$Prob[A_v^+] \leq 2e^{-2\delta^+(v)^{\frac{1}{3}}} \tag{4}$$

Similarly,

$$Prob[A_v^-] \leq 2e^{-2\delta^-(v)^{\frac{1}{3}}} \tag{5}$$

We are given $\delta^+(v) \geq x$, and $\delta^-(v) \geq x$. Hence,

$$\begin{aligned} Prob[A_v^+] &\leq 2e^{-2x^{\frac{1}{3}}} \\ Prob[A_v^-] &\leq 2e^{-2x^{\frac{1}{3}}} \end{aligned}$$

How many events might A_v^+ depend upon? It is mutually independent of all but at most

$$\sum_{u\in N^+(v)} (\delta^+(u) + \delta^-(u) - 1) \leq \sum_{N^+(v)} (8x - 1) \tag{6}$$

Similarly for A_v^- (A_v^- can be shown to be independent of all but at most $32x^2 - 1$ other events).

We now have to verify $e \cdot p \cdot (d+1) \leq 1$ to use L^3. That is,

$$e \cdot 2e^{-2x^{\frac{1}{3}}} \cdot 32x^2 \leq 1 \tag{7}$$

which holds for $x \geq 1000$. $\square$

Recall Lemma 1. The proof is by repeated application of Lemma 2.