22C:253 Seminar on Randomization 22C:296

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Chernoff Bounds and Applications

Chebysev's inequality does give a concentration result, but it is very weak. In many applications, we need stronger results. For example, Let $X_1, \cdots, X_n$ be binary random variables. Let $P[X_i = 1] = p_i$, Let $X = \sum_{i=1}^{n} X_i$.

Suppose the X_i's are all mutually independent, from Chebysev's inequality we get that

$$Pr[|X - E[X]| > t] \leq \frac{Var[X]}{t^2}$$

In our case,

$$\begin{aligned} E[X] &= \sum p_i = \mu \\ Var[X] &= \sum Var[X_i] = \sum_{i=1}^{n} p_i(1-p_i) \end{aligned}$$

Hence, we get

$$P[|X - E[X]| \geq t] \leq \frac{\sum_{i=1}^{n} p_i(1-p_i)}{t^2}.$$

To simplify this, let the X_i's be identical, i.e., $p_i = \frac{1}{n}$, for all $i = 1, \cdots n$. Then,

$$P[|X - \mu| \geq t] \leq \frac{n}{4t^2}.$$

If $t = \delta\mu$, then the above inequality becomes

$$P[|X - \mu| \geq \delta\mu] \leq \frac{n}{4\delta^2\mu^2} = \frac{1}{\delta^2 n}.$$

This probability approaches 0 as $n \to \infty$ but does so only linearly.

For this very example, using Chernoff bounds, we can show an upper bound that grows exponentially as $n \to \infty$.

The other reason it is important to show such a tight bound is that often the Chernoff bound is used before applying the union bound on an exponential number of events. We should still be able to conclude that the total probability is < 1.

Chernoff Bounds : As before, let $X_1, \cdots X_n$ be binary random variables such that $P[X_i = 1] = p_i$, and let $X = \sum_{i=1}^{n} X_i$.

Suppose the X_i's are mutually independent, then the Chernoff bound says that :

$$P[X > (1+\delta)\mu] < \Big(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\Big)^{\mu}$$

Example : Suppose the X_i's are all identical, and $P[X_i = 1] = \frac{1}{n}$, then by Chernoff bounds, we get

$$P[X > (1+\delta)\mu] < \Big(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\Big)^{\frac{n}{2}}$$

Then, for $\delta > 2e - 1$, we get

$$P[X > 2e\mu] < \Big(\frac{e^{2e}}{(2e)^{(2e)}}\Big)^{\frac{n}{2}} = \Big(\frac{e}{2e}\Big)^{n} = \Big(\frac{1}{2}\Big)^{n}$$

Proof of Chernoff Bounds :

For any positive t, $X > (1+\delta)\mu = e^{tx} > e^{t(1+\delta)\mu}$.

Therefore,

$$P[X > (1+\delta)\mu] = P[e^{tx} > e^{t(1+\delta)\mu}]$$

by Markov's inequality. The RHS is bounded above by

$$\begin{aligned} P[X > (1+\delta)\mu] &< \frac{E[e^{tx}]}{e^{t(1+\delta)\mu}} \\ &= \frac{E[e^{t\sum_{i=1}^{n} X_i}]}{e^{t(1+\delta)\mu}} \\ &= \frac{E[\Pi_{i=1}^{n} e^{tX_i}]}{e^{t(1+\delta)\mu}} \\ &= \Pi_{i=1}^{n} \frac{E[e^{tX_i}]}{e^{t(1+\delta)\mu}} \text{ by mutual independence of } X_i\text{'s} \end{aligned}$$

$E[e^{tX_i}] = (1 - p_i) + e^t p_i$.

Therefore,

$$P[X > (1+\delta)\mu] \leq \Pi_{i=1}^{n} \frac{1 + p_i(e^t - 1)}{e^{t(1+\delta)\mu}}$$

Now, using the fact that $e^x > (1 + x)$, for $x \neq 0$,

we get that

$$
\begin{aligned}
P[X > (1+\delta)\mu] &> \Pi_{i=1}^{n} \frac{e^{p_i(e^t-1)}}{e^{t(1+\delta)\mu}} \\
&= \frac{e^{(t-1)\mu}}{e^{t(1+\delta)\mu}}
\end{aligned}
$$

To get the tightest possible upper bound, we find a t that minimizes the RHS. This we get by differentiating the RHS. The minimal value of t thus found is $t = ln(1+\delta)$. Now substituting, we get the desired result.

Remark : A similar result can be shown for $P[X < (1-\delta)\mu]$.

In this case, the Chernoff bound gives us :

$$P[X < (1-\delta)\mu] < \frac{e^{-\delta}}{e^{-\delta+\delta^2/2}}$$

Applications :

- Oblivious permutation routing in a network
- Existence of a edge-disjoint cycles in d-regular digraphs (using the *semi-random* technique, or Rodl's nibble).