# 22C: 253 Lecture 9

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Last class we presented a factor-1/2 approximation algorithm for MAX-SAT. Now our goal is to improve this to a factor-3/4 approximation algorithm. Here is our factor-1/2 algorithm:
**Algorithm 1**: Set each variable $x_i$ to TRUE independently, with probability 1/2.
Our next algorithm uses LP relaxation followed by randomized rounding.
**Algorithm 2 (Randomized Rounding Algorithm)**
Start with an IP for MAX_SAT. Let $z_C$ be an indicator variable indicating if clause $C$ is *TRUE* or *FALSE*. For each clause $C$, let $L_C^+$ denote the set of positive literals in $C$, and $L_C^-$ denote the set of negative literals in $C$.

$$\begin{aligned}
&\max \sum_C w_C z_C \\
&\textit{subject to} \\
&z_C \leq \sum_{i \in L_C^+} x_i + \sum_{i \in L_C^-} (1 - x_i) \\
&z_C \in \{0,1\} \text{for each clause } C \\
&x_i \in \{0,1\} \text{for each } i = 1, 2, \ldots, n
\end{aligned}$$

Let $x_i = 1$ denote setting of $x_i = \textit{TRUE}$ and $x_i = 0$ denote setting of $x_i = \textit{FALSE}$. In the corresponding LP-relaxation, we replace
$z_C \in \{0,1\}$ by $0 \leq z_C \leq 1$, and $x_i \in \{0,1\}$ by $0 \leq x_i \leq 1$.

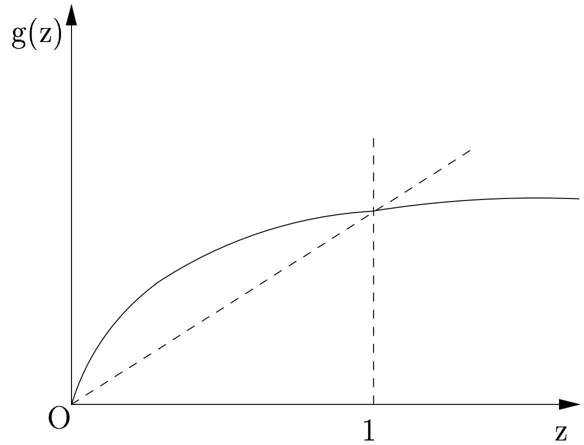
**Randomized Rounding Algorithm**

**Step 1:** Solve the LP-relaxation and let $(x^*, z^*)$ denote an optimal solution.

**Step 2:** For each $i = 1, 2, 3, \ldots, n$, set $x_i = TRUE$ with probability $x_i^*$, and $x_i = FALSE$ with probability $(1 - x_i^*)$.

Let us analyze this algorithm. Pick an arbitrary clause $C$ and suppose it has $k$ literals. Without loss of generality, assume

- The literals in C involve distinct variables.
- The literals in C are all positive.
- $C = (x_1 \vee x_2 \vee x_3 \vee \cdots lor x_k)$.




Then Prob[$C$ is FALSE] $= \prod_{i=1}^{k}(1-x_i^*)$. It is a fact that for nonnegative numbers of $a_1, a_2, \ldots, a_k$, the arithmetic mean is at least as large as the geometric mean. In other words,

$$\frac{a_1 + \cdots + a_n}{k} \geq \sqrt[k]{a_1 a_2 \cdots a_k}.$$

This implies that

$$\text{Prob}[C \text{ is FALSE}] \leq \Big(\sum_{i=1}^{k} \frac{(1-x_i^*)}{k}\Big)^k$$

Since $x^*$ is feasible for the LP-relaxation, it satisfies

$$\sum_{i=1} kx_i^* \geq z_C^*.$$

Hence,

$$\text{Prob}[C \text{ is FALSE}] \leq (1 - \frac{z_C^*}{k})^k$$

and this implies that

$$\text{Prob}[C \text{ is TRUE}] \leq 1 - (1 - \frac{z_C^*}{k})^k.$$

We need to understand the function $g(z) \leq 1 - (1 - \frac{z}{k})^k$ better to take the next step.Suppose $\beta_k = 1 - (1 - frac1k)^k$.

**Lemma 1** *$g(z) \geq \beta_k z_k$, for $z \in$[0,1].*

**Proof:**

$$g'(z) = -k(1 - \frac{z}{k})^{k-1}(-\frac{1}{k}) = (1 - \frac{z}{k})^{k-1}$$

$$g''(x) = (k-1)(1 - \frac{z}{k})^{k-2}(-\frac{1}{k}) < 0$$

This implies that $g'(z)$ is decreasing and the function looks as shown in the figure above. So $g(z) \geq \beta_k z$ for $z \in [0,1]$. $\square$ This implies that Prob[$C$ is TRUE] $\geq \beta_k z_C^*$. Therefore $E[W_C] \geq \beta_k z_C^* w_C$. We know $\beta_k = 1 - (1 - \frac{1}{k})^k \geq 1 - \frac{1}{e}$, and therefore

$$E[W_C] \geq (1 - \frac{1}{e}) z_C^* w_C.$$

This implies that

$$E[W] \geq (1 - \frac{1}{e}) \sum_C z_C^* w_C \geq (1 - \frac{1}{e}) OPT.$$

Let us reexamine the analysis of the two algorithms. Let C be a clause with k literals,
**Algorithm 1:** Prob[$C$ is TRUE] $= 1 - \frac{1}{2^k} = \alpha_k$.
**Algorithm 2:** Prob[$C$ is TRUE] $\geq \beta_k z_k^* = (1 - (1 - \frac{1}{k})^k) z_C^*$.

| | $k$=1 | $k$=2 | $k$=3 | |
|---|---|---|---|---|
| $\alpha_k$ | 1/2 | 3/3 | 7/8 | $\alpha_k$ is an increasing function of k $\Rightarrow$ so algorithm 1 does well for large clauses. |
| $\beta_k$ | 1 | 3/3 | 19/27 | $\beta_k$is a decreasing function of k $\Rightarrow$ algorithm 2 does poorly for large clauses. |

It is also easy to verify that $\alpha_k + \beta_k \geq 3/2$ for all $k$. This suggests a third algorithm that performs better by picking one of Algorithm 1 or Algorithm 2, randomly.

**Algorithm 3:** Toss a coin and run algorithm 1 or algorithm 2 depending on the outcome.

**Lemma 2** $E[W] \geq \frac{3}{4} OPT$.

**Proof:** Let $W_1$ and $W_2$be the random variables denoting weight of solution of algorithm 1 and algorithm 2 respectively. Let C be a clause with $k$ literals. Let $W_C^1$ and $W_C^2$ denote the random variable that stands for the weight contribution of clause $C$ for algorithm 1 and algorithm 2 respectively. We know $E[W_C^1] = \alpha_k w_C$ and $E[W_C^2] \geq \beta_k z_C^* w_C$. Let $W_C$ be the weight combination of clause $C$ in combined algorithm. Then,

$$W_C = \begin{cases} W_C^1 \text{ with probability } 1/2 \\ \\ W_C^2 \text{ with probability } 1/2 \end{cases}$$

Hence,

$$E[W_C] = (W_C^1 + W_C^2)/2.$$

By substituting the bounds for the individual algorithms we get

$$E[W_C] \geq \frac{1}{2}(\alpha_k w_C + \beta_k w_C z_C^*).$$

Since $z_C^* \leq 1$, this implies

$$E[W_C] \geq \frac{1}{2}(\alpha_k w_C z_C^* + \beta_k w_C z_C^*).$$

Finally,

$$E[W_C] \geq \frac{1}{2}(\alpha_k + \beta_k) w_C z_C^* \geq \frac{3}{4} w_C z_C^*.$$

Therefore, $E[W] = \sum_C E[W_C] \geq \frac{3}{4} \sum_C w_C z_C^* \geq \frac{3}{4} OPT.$ $\square$