22c253 Algorithms in discrete Opt

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November 14, 2002

From F_t get I, which are permanently open facilities by finding maximal independent set. Then we have to determine the connection ϕ.

Consider a city $j \in C$, $Y_j = \{i \in F | B_{ij} > 0\}$.

1. If $|X_j \cap I| = 1$ then set $\phi(j) = i$ where $i \in X_j \cap I$.

2. Otherwise, let i' be the connecting witness for j. $i' \in I$ then set $\phi(j) = i'$.

3. Otherwise (i.e. $i' \in I$) there is a neighbor (in H) of i' in I. Call this neighbor i'' and set $\phi(j) = i''$.

Claim: The dual feasible solution (α, β) and the integral primal feasible solution (I, ϕ) satisfy the slackness condition listed.

Proof: Suppose that y_i and x_{ij} denote the feasible solution after phase 2.

(1) y_i implies $i \in I$. Since $I \in F_t$ and F_t only contain facility $i \in F$ s.t. $\sum \beta_{ij} = F_t$ and furthermore, since β_{ij} do not increase once $i \in F_t$. Therefore this is true.

(3) is trivial

(4) Suppose that for some $i \in F, j \in C : \beta_{ij} > 0$. If $y_i = 0$, then clearly $x_{ij} = 0$.

If $y_i = 1$, then is must be the case that for any $i' \in F$ s.t. $\beta_{ij} > 0$, then $i' \notin I$.

$\Rightarrow \phi(j)$ is set to i in step 1. Therefore $x_{ij} = 1$.

(2) Consider $i \in F, j \in C : x_{ij} > 0$. This implies that $x_{ij} = 1$.

(i.e.) city i is the connected to facility j.

This connection can be in (1˜3) of the steps defining ϕ. If the connections made in (1) then $\alpha_{j-}\beta_{ij} = C_{ij}$. When the connection is made in (3), there are some other cities that are making positive contribution to i'. We need to show that : $\alpha_j \geq C_{ij}/3$.

(Figure 1 comes into this part.)

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Assume that we increase α_j simultaneously, say that i was thrown into F_t at time t_1 and i' and t_2.

Claim: $\alpha_i \geq t_2 \Rightarrow \alpha_j \geq \alpha_{j'}$.

$\alpha_j \geq C_{i'j}$ because $\alpha_j - \beta = C_{ij'}$. $\alpha_{j'} > C_{ij'}$. These imply $\alpha_j > C_{i'j'}$ and $\alpha_j > C_{ij'}$.

Also $\alpha_j \geq C_{i'j} \Rightarrow 3\alpha_j \geq C_{ij} + C_{i'j} + C_{i'j'} \geq C_{ij}$.

Therefore $\alpha_j \geq C_{ij}/3$.

Next topic is approximation algorithm using Semidefinite-Programming by goemans and Williamson 1995.

Example: Max-Cut

Imput: A graph $G = (V, E)$

Outout: A partition of V into $(S, \overline{S})$ s.t. sum of crossing edge weights is maximized.

Suppose for each vertex $i \in V$, we have a variable $y_i \in \{\pm 1\}$, where $y_i = 1$ if $i \in S$. $y_i = -1$ otherwise. And $S \cup \overline{S} = V$.

$\rightarrow i, j$ in the same set then $y_i y_j = 1$ and $(1 - y_i y_j)/2 = 0$.

$\rightarrow i, j$ not in the same set then $y_i y_j = -1$ and $(1 - y_i y_j)/2 = 1$.

We would like to solve the following program.

max $\sum_{(i,j)} \frac{(1-y_i y_j)}{2} w_{ij}$ where $y_i \in \{i, j\}$ or $y_i^2 = 1$. This is a strict quadratic program. To relax this problem, we solve

max $\sum_{(i,j)} \frac{(1-v_i v_j)}{2} w_{ij}$ where v_i are n-dimensional vectors s.t. $v_i v_i = 1$.