Lecture Notes: Social Networks: Models, Algorithms, and Applications
Lecture 8: Feb 9, 2012
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1 Watts-Strogatz Model

For a positive integer n and an even integer k, let $C(n,k)$ denote the graph with vertex set $\{0,1,2,...,n-1\}$ and edge set $\{\{i,j\}, 0 \leq i,j \leq n-1, |i-j| \leq k/2\}$.

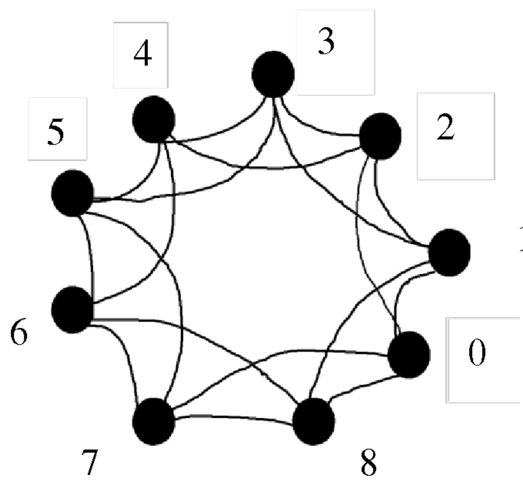

Fig. 1

Figure 1: $C(8,4)$, newly rewired edges are excluded from future rewire.

The Watts-Strogatz graph[2], denoted $WS(n,k,p)$ is obtained from $C(n,k)$ by replacing each edge in $C(n,k)$ with probability p by a randomly chosen edge.

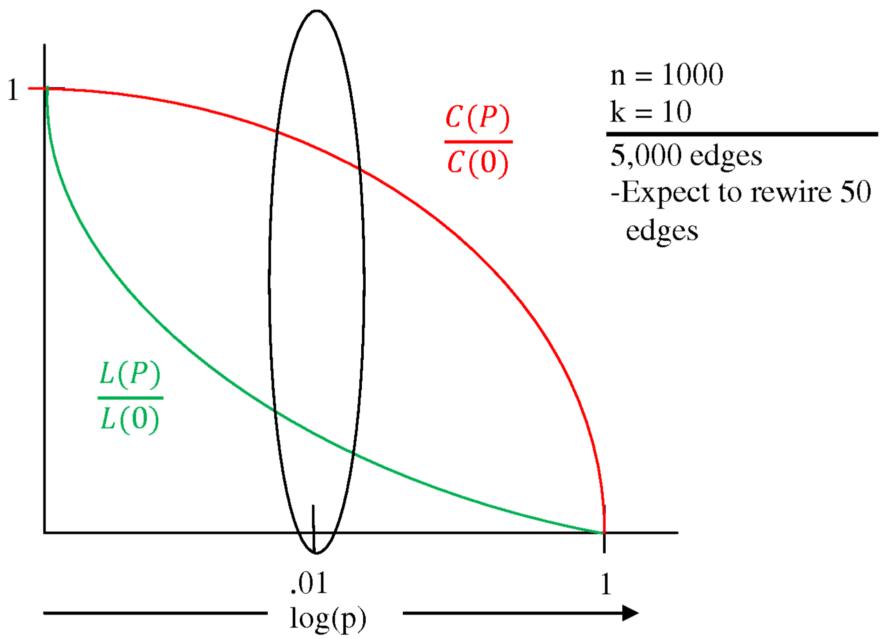

Fig. 2

Figure 2: $C(P)$ represents the expected clustering coefficient of $WS(n,k,p)$. $L(P)$ represents the expected average path length of $WS(n,k,p)$. It is shown that every 20 vertices receives a rewire of edges.

2 Discussion

Proximity. The base graph $C(n,k)$ starts by connecting vertices that are close by. For example, $Grid(n,r)$: vertex set $\{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-1\}$ and edge set $\{(i_1,j_1),(i_2,j_2)\}$, where $|i_1 - i_2| + |j_1 - j_2| \leq r$. An example of $Grid(4,2)$ is shown in Fig. 3.

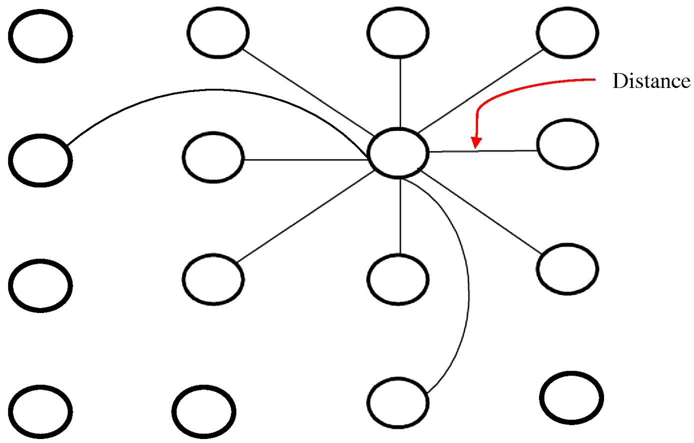

Fig. 3

Figure 3: $Grid(4,2)$

It can be more abstract. Let $M = (V, d)$ be a metric space. Consider the graph with vertex set V and edge set: $\{u, v : d(u, v) \leq r\}$, where r is some parameter.

Randomness is added in a variety of ways to achieve the same effect. Alternative Approach: start with $C(n, k)$. To each vertex u, add an edge $\{u, v\}$ with v chosen randomly.

Result. This result made a lot of sense to sociologists because they believed in two types of edges:

1. edges induced by homophily $\Rightarrow$ base graph edges
2. edges that correspond to weak ties $\Rightarrow$ random edges

$$\text{homophily} + \text{weak ties} \Rightarrow \text{small world property.}$$

Recall.

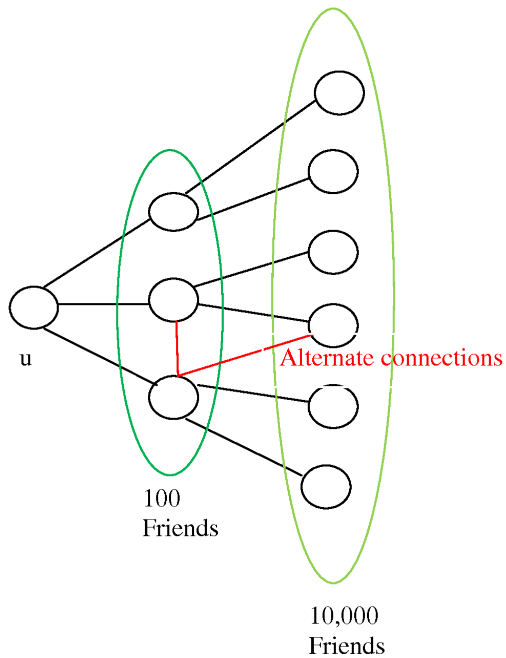

Fig. 4

Figure 4: Alternate edges/connections may dampen the size of the set and may elongate the graph. Alternate edges/connections can also make the lengths to other edges quicker.

$$\text{Clustering Coefficient}^{1} \uparrow \Rightarrow \text{Average Path Length} \uparrow$$

Diameter. The Diameter of a Cycle Plus a Random Matching[1]. See Fig. 5. for example.

[1]Node based definition of clustering coefficient.

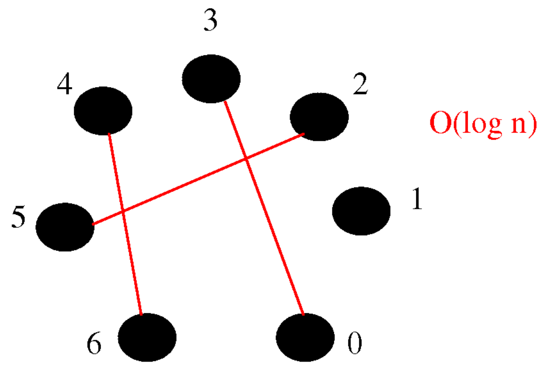

Fig. 5

Figure 5:

3 Kleinberg's Question

Watts-Strogatz model[2] is small world. Does it also allow efficient decentralized (local) search?

Example. Condider $WS(n, k, p)$. Let s and r denote sender and receiver. The sender, knowing only r's label, has a package that needs to be sent to r. Typical Step: node v on receiving the package, either:

1. If v has a neighbor closer to r than itself, v sends the package to the neighbor closest to r.
2. Otherwise, v gives up.

Questions. Suppose we pick s and r randomly and perform graphic greedy routing many times:

- What fraction of these experiments is successful?
- What is the average path length of the successful experiments?

4 Kleinberg's Model[3]

Let us use $K(n, r, q, -\alpha)$ to denote the graph obtained by starting with $Grid(n, r)$ and adding random edges as follows: To each vertex u, add q random edges $\{u, v\}$ with v picked out with a probability proportional to $d(u, v)^{-\alpha}$. If $\alpha = 2$, then $d(u, v)^{-\alpha} = 1/d(u, v)^2$. Fig. 6 demonstrates the model.

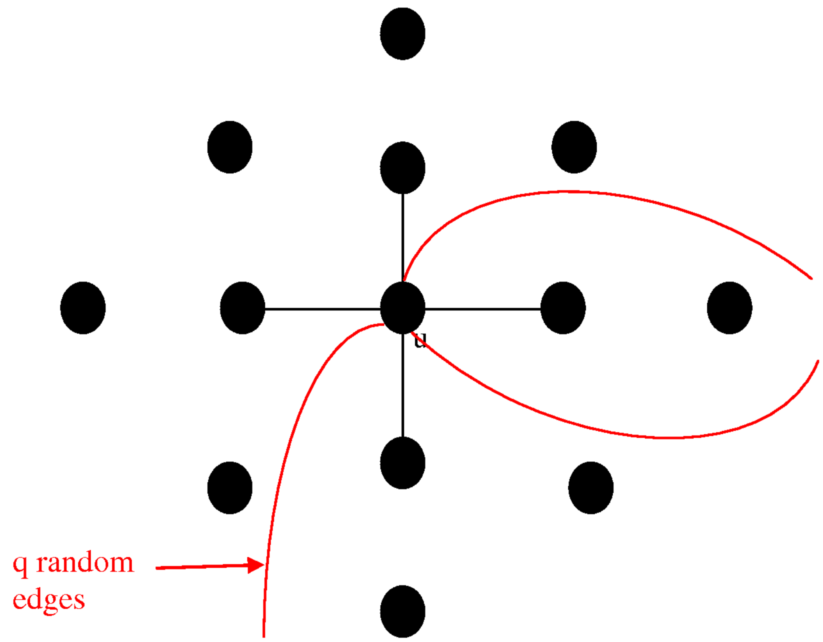

Fig. 6

Figure 6: One hop edges have higher probability to be connected than 2 or more hop edges. If $\alpha = 1$, then the probability distribution is uniform such that no differentiation between near neighbors and far nodes.

Results.

1. For $\alpha = 0$, any decentralized algorithm requires at least $(n^{2/3})$hops
2. For $\alpha = 2$, geographic greedy routing discovers paths of expected length $O(\log 2n)$

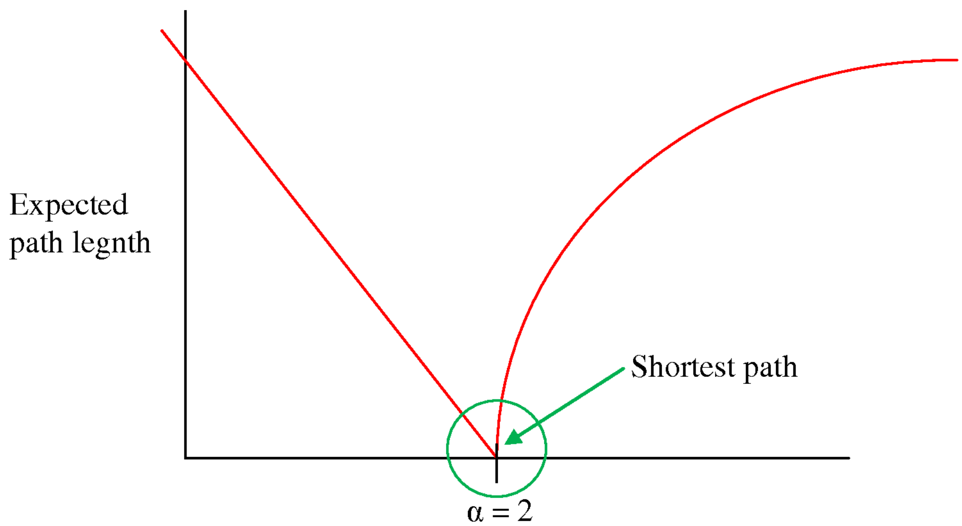

Figure 7: The correlation between α and the expected path length.

References

[1] B. Bollobs and F. R. K. Chung. The diameter of a cycle plus a random matching. *SIAM J. on Discrete Mathematics*, 1(3):328–333, 1998.

[2] Duncan J.Watts and Steven H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:440–442, 1998.

[3] J. Kleinberg. Navigation in a small world. *Nature*, 406:845, 2000.