Lecture Notes: Social Networks: Models, Algorithms, and Applications
Lecture 6: Feb 2, 2012
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1 Proof Continued from the Previous Class

Prob[X > (1 + δ) μ] < $e^{\frac{-\mu\delta^2}{4}}$ when,$\delta \leq$ 2e-1

Now, μ = C|S_i|

$\delta = \frac{1}{4}$

So, plugging the values in the bound provide us with the following equation:

$e^{\frac{-C|S_i|\cdot\frac{1}{n}^2}{4}} = e^{\frac{-C|S_i|}{64}}$

[Ensures every time that the elements are independent]

Now, p(n) $\geq$ C$\cdot\frac{\ln n}{n}$ where, C is contant

So, C= p(n)(n-1) $\geq$ C$\cdot$ln n
Plugging this into the bounds, we get an upper bond of

$e^{\frac{-C \ln n|S_i|}{64}}$ As, $e^{\ln} = \frac{1}{n}$

Picking C large enough gives a bound $(\frac{1}{n})^{|S_i|} \leq \frac{1}{n}$

By this upper bound is bound as $(\frac{5C}{4})|S_i| \leq \frac{1}{n}$

So, Prob[$|S_{i+1}| > \frac{5C}{4}$ $|S_i|$] $\leq \frac{1}{n}$

Simmilarly, Prob[$|S_{i+1}| < \frac{C}{4}$ $|S_i|$] $\leq \frac{1}{n}$ $\square$

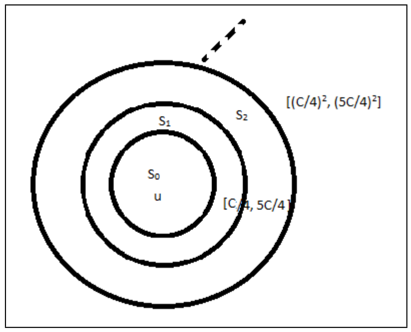
Lemma Let, $|T_i| \geq \frac{n-1}{2}$ then,

Prob[$\frac{C}{4}^{i+1} \leq |S_i + 1| \leq \frac{5C}{4}^{i+1}$] $\geq$ 1- $\frac{1}{n}$

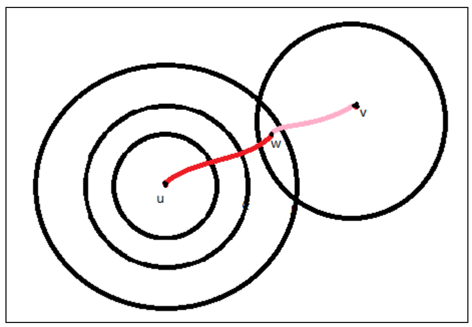
This will be true, only if S_0, S_1,......, S_{i+1} follow this rule.Now as we know, this is true with a very high probability. As the probability of its non-occurence is only $\frac{2(i+1)}{n}$. With changes in the value of C[constant] this probability becomes $\frac{2}{n}$, which is very small.

Theorem Let u,v $\in$ V. Then, with probability $\geq$ 1- $\frac{1}{n}$

d(u,v) $\leq$ O($\frac{\ln n}{\ln C}$)



This goes on until we reaches $|T_i| < \frac{n-1}{2}$
Now, as we reached to the end set, say S_{i+1} then the ball having the reached nodes strictly has more than half of the nodes of the graph.



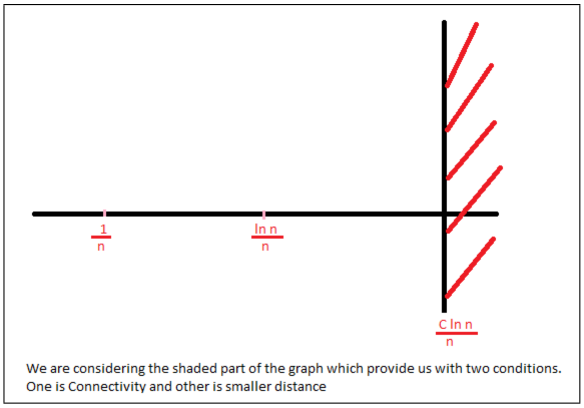
So, d(u,w) = $\frac{\ln n}{\ln C}$

And, d(w, v) = $\frac{\ln n}{\ln C}$

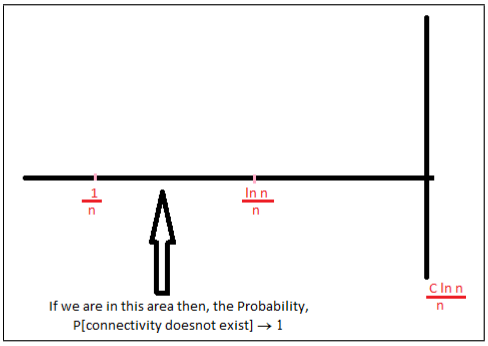
The above relation was proved assuming, p(n) $\geq$ C$\frac{\ln n}{n}$

But, the claim is actually true for, p(n) $> \frac{1}{n}$

Now, a big problem occurs due to this. As, these two points occur at different times.



2 Phase Transition In ER Graphs



Lemma If, p(n) < $\frac{\ln n}{n}$ then with probability → 1 as n → ∞, the graph has atleast one isolated vertex. This shows disconnectivity of the graph.

Proof For a vertex u ∈ V, let:

$$I_u = \begin{cases} 1 & \textit{if u is isolated} \\ 0 & \textit{otherwise} \end{cases}$$

Prob[I_u = 1] = $(1\text{-p})^{\text{n-1}} \sim e^{\text{-p(n-1)}}$

So, E[I_u] $\sim e^{\text{-p(n-1)}}$

Let, X= $\sum I_u = ne^{\text{-p(n-1)}}$

Suppose, p = $\lambda\frac{\ln n}{n}$ where, $\lambda < 1$

then, E[X] = $ne^{-\lambda \ln n} = n^{1-\lambda}$

if, $\lambda = 0.9$, then $n^{0.1}$ is expected isolated graph.

Note that, As n → ∞, E[X] → ∞
Further calculation involving the variance can be used to show that the,
Prob[there exist an isloated vertex] → 1 as n → ∞ □

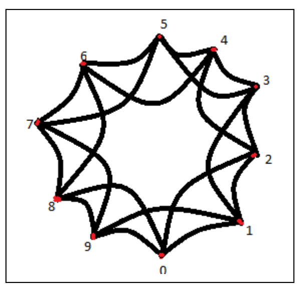
3 Watts Strogatz Model

The table below describes the various networks with their degree and cluster coefficient.[1]

	N	Average degree	CC	CC of corresponding ER Graph
Actor Network	225,226	61	0.79	0.00027
Power Grid	4941	2.67	0.080	0.005
C.elegance	282	14	0.28	0.05

There are observed networks that show the properties like:
1. Sparse(average degree is small relative to N)
2. Small average path length relative to N
3. Cluster coefficient is highrelative to that of corresponding ER graph.

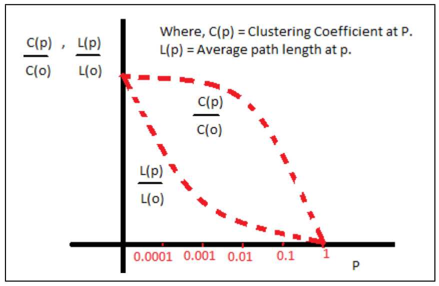
Ques) Is there a simple random graph model with these 3 characteristics?
Answer) If we take a Circular Graph : C(n,k)
for n= 10, k=4, C(10,4) is like:



Then, the above discussed points are satisfiable as:

1. Sparsity is controlled by k, if k is large, sparsity is less.
2. Cluster coefficient is high, as every V-node is connected to the neighbours, half on one side and half on other side, and these neighbours are also connected in simmilar way.
3. Average path length depends on k, for high value of k, it will be small.

Watts Strogatz Model WS(n,k,p) where $0 \leq p \leq 1$
Now, we choose 2 vertex and one edge and reconnect it,
As, p goes from 0 to 1, randomness of the graph increases.
At, p = 1, Original graph is completely lost and we have a totally random graph ER(n, ?)
At, p = 0, we have an original graph C(n,k).
In intermediate region the property of graph is like :



So, at very small randomness, we see a lot of decrease in the path length but we do not see a lot of change in clustering coefficient.

References

[1] D. J. Watts and S. H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393(6684):440–442, June 1998.