Notes for 22c196

May 3, 2012

1 Matching

We want to extend the matching algorithms to a decentralized solution. We do this by turning the matching problem into a market.

We begin with a set B of n buyers and a set S of n sellers. Each buyer $i \in B$ has, for each seller $j \in S$, a valuation v_{ij} of that seller's item to that buyer. If we have the situation where $|B| \neq |S|$, we can fabricate additional sellers or buyers as necessary, setting $v_{ij} = 0$ if either the associated seller or buyer was added.

Let p_i be the price (to be determined) for which item j is sold. If bidder i wins this item, they will have a payoff of $v_{ij} - p_i$.

A *prefered seller* is one that maximizes the payoff for a given buyer. Formally, the set $Pref_i = \{j \mid \arg\max_j v_{ij} - p_i\}$.

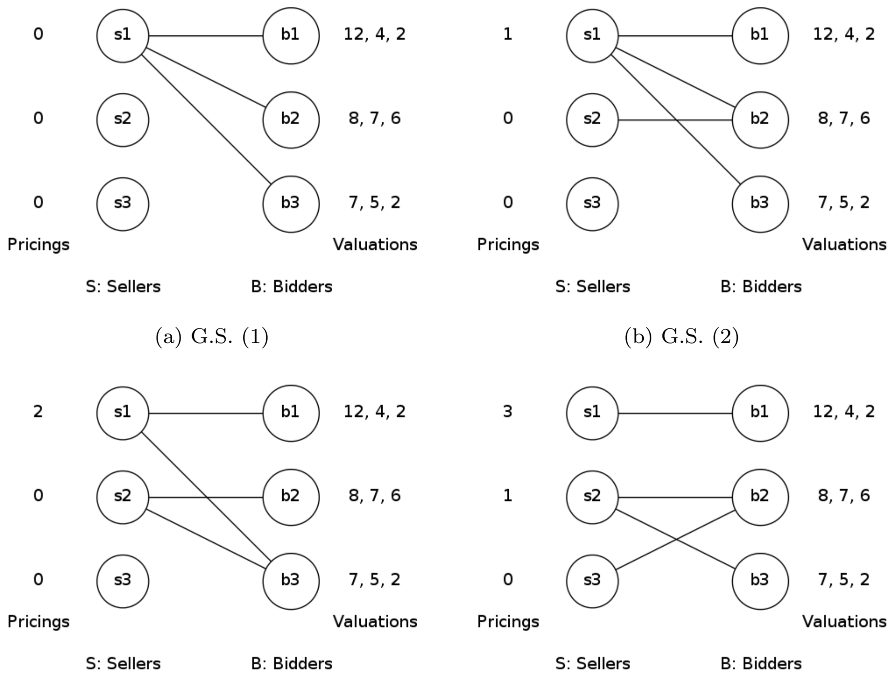
We consider the bipartite graph of buyers and sellers, where an edge exists between a buyer and its prefered sellers. A set of *market-clearing prices* are prices that yield a perfect matching in the prefered sellers graph. If a set of market-clearing prices exists, it will maximize the social good; each buyer will be assigned to their 'favorite' seller, and each seller can sell their item for maximal revenue.

Surprising Statement 1. *For any any set of buyer valuations, there exists a set of market clearing prices.*

Surprising Statement 2. *For any set of market-clearing prices, any resulting perfect match yields maximum total valuation, which we define as* $\sum_j [v_{ij} - p_i]$.

We note that the sum above can be expanded to $\sum_j v_{ij} - \sum_j p_i$. We note that this value is independent of the particular choice in perfect matching, in cases where multiple exist. This means that any perfect matching maximizes this value. Buyers will be made "maximally happy," and sellers maximize their prices. In effect, the most possible money is changing hands.

We would like to *determine* a set of market-clearing prieces, given the valuations of the buyers. If we have global knowledge of all valuations, we can employ the augmented path algorithm discussed previously or some other prefect matching algorithm. However, we would not expect such an aboundance of information in real world situations. We examine the Gale-Shapely matching algorithm to set market-clearing prices using only local information. We can also refer to Demage, Gale, and Sotomajor (1986) and Egerváry (1916) for more information.



(a) G.S. (1)

(b) G.S. (2)

(c) G.S. (3)

(d) G.S. (4)

Figure 1: Gale Shapeley Algorithm Example: In its final step, we have $\{(1,1),(2,3),(3,2)\}$ as our perfect matching.

Gale-Shapely Algorithm

(1) All sellers set their prices to zero.

(2) Buyers choose their prefered sellers.

(3) Does a perfect matching exist? If yes, we exit, accepting the current set of prices.

(4) Since a perfect matching does not exist, there must be some constriced set. We find the constricted set(s).

(5) Each seller in a constricted set increases the cost of their item by one.

(6) If $\min_j p_j > 0$, each seller reduces their price by $\min_j p_j$.

(7) Go to (2).

(See Figure 1 for an example execution.)

Proff of Termination in Gale-Shapely

We consider two payoff potentials; define

$$P_b = \sum_j max_i(v_{ij} - p_i)$$

as the maximum payoff potential for buyers and

$$P_s = \sum_i p_i$$

as the maximum payoff for sellers.

Consider $P_s + P_b$ as a utility function. At each iteration, the program will either terminate in step (3), or we have a constricted set. Let S denote the constricting set, and let $N(S)$ denote the union of all neighbors of S. P_s will increase by $|N(S)|$, and P_b will decrease by $|S|$. But by definition of the constricted set, $|S| > |N(S)|$, so $P_b + P_s$ decreases. Additionally, $v_{ij} \geq p_i \geq 0$ for all i, j, so $P_s + P_b$ is bounded below by zero. Also, step (6) leaves the sum unchanged.

Since the utility is decreasing at every step and bounded below, we know the algorithm will complete in finite time.

2 Per-Click Adertisement

Suppose we have some number of advertisement slots (sellers) and some number of advertisers (buyers). Again, we assume that the number of sellers and the number of buyers are equal; if they are not, we add sellers of value zero or buyers with zero wallet, as appropraite.

This problem could be approached as before, wherein we assign a fixed price to each slot. v_{ij} may denote the expected revenue from buyer j using slot i. However, and advertiser might not want to reveal their valuations. We need a mechanism that maximizes social good (that is, the buy obtains the best value and the seller makes profitable sales), but without revealing valuations. For these, we have auctions.

The *Vickery, Clark, Groves principle* (VCG) is an auction design that can be thought of as a generalization of Second Price Auctions, where truth telling remains the dominating strategy. In auctions with VCG, the winning bidder is charged a price determined by the "harm" dome to the overall social good.

Consider a collection of bids $b_1 > b_2 > \cdots$ for some Second Price Auction. If bidder 1 has not been a part of this auction, the only person harmed is bidder 2; all other bidders would still lose. The harm done by bidder 1 is the payoff bidder 2 consequently lost.

In general, if S is the set of sellers and B is the set of Buyers, then V_B^S is the maximum total valuations over all perfect matchings, that is, the socially optimal outcome. Note that if item i is assigned to buyer j, then $V_{B\setminus\{j\}}^{S\setminus\{i\}}$ is the maximum valuation for everyone else.

It follows that, if bidder j wins item i, the they are responsible for social harm given by

$$V_{B\setminus\{j\}}^{S} - V_{B\setminus\{j\}}^{S\setminus\{i\}}$$

We establish the following protocol:

- Buyers bid on slots (they needn't tell the truth).
- Find a set of market clearing prices.
- Charge bidders VCG prices, that is, $p_{ij} = V_{B\setminus\{j\}}^{S} - V_{B\setminus\{j\}}^{S\setminus\{i\}}$

We claim that this protocol yields a dominating strategy of truth telling, and that the market clearing prices maximize total value.

If this protocol does, in fact, yield a dominating strategy of truth telling, the second part is easy; if all bids were honest, the market clearing prices provide the socially optimal output, by their definition.

3 Dominating Strategy in VCG Auctions

Suppose that when buyer j bids honestly, she wins item i. The payoff to buyer j for winning item i is $v_{ij} - p_{ij}$. Notice that the prices are personalized. Furthermore, $p_{ij} == V_{B\setminus\{j\}}^{S} - V_{B\setminus\{j\}}^{S\setminus\{i\}}$ depends on the other buyers' bids. As such, if bidder j bids dishonestly and still wins item i, the payoff for bidder j is unchanged.

Suppose instead that bidder j bids dishonestly and subsequently wins some other item h. The payoff for bidder j is now $v_{hj} - p_{hj}$. To benifet from this lie, bidder j needs

$$v_{hj} - p_{hj} > v_{ij} - p_{ij}$$

to hold. If, instead, we show that $v_{hj} - p_{hj} \leq v_{ij} - p_{ij}$, then the above inequality does not hold and there is no incentive for bidder j to bid dishonestly.

$$\begin{aligned}
v_{hj} - p_{hj} &\overset{?}{\leq} v_{ij} - p_{ij} \\
v_{hj} - \left(V_{B\setminus\{j\}}^{S} - V_{B\setminus\{j\}}^{S\setminus\{h\}}\right) &\overset{?}{\leq} v_{ij} - \left(V_{B\setminus\{j\}}^{S} - V_{B\setminus\{j\}}^{S\setminus\{i\}}\right) \\
v_{hj} + V_{B\setminus\{j\}}^{S\setminus\{h\}} &\overset{?}{\leq} v_{ij} + V_{B\setminus\{j\}}^{S\setminus\{i\}}
\end{aligned}$$

Now, we know that V_B^S is both socially and individually optimal. We had assumed that bidder j is assigned item i when she bids honestly. As such, $V_{B\setminus\{j\}}^{S\setminus\{i\}}$ is the same assignment to everyone, exclusind item i and bidder j. However, if these assign j to i optimally, it follows that any assignment of j to item h

is suboptimal. Thereofre, $V_{B\setminus\{j\}}^{S\setminus\{h\}} \leq V_B^S = V_{B\setminus\{j\}}^{S\setminus\{i\}}$ and $v_{hj} < v_{ij}$. It follows that $v_{hj} + V_{B\setminus\{j\}}^{S\setminus\{h\}} \leq v_{ij} + V_{B\setminus\{j\}}^{S\setminus\{i\}}$, concluding the proof and showing that honest bidding is the dominating strategy.

It should be noted that, while this policy maximises *social* wellfare, Google uses a "Generalized Second Price Auction". Google isn't interested in social welfare; they want to maximize the seller's happiness.