Lecture Notes: Social Networks: Models, Algorithms, and Applications
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1 Game-theoretic Modeling of Traffic Congestion

Consider the following algorithm that converges to an N.E.[1]:

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GMTG(G, (p_1, p_2, ..., p_n))
    1.  start with an arbitrary strategy choice (p_1, p_2, ..., p_n).
    2.  while (p_1, p_2, ..., p_n) is not a N.E.:.
    3.  pick p_i that isnt the best response to the remaining choices
            (p_1, p_2, ..., p_{i-1}, p_{i+1}, ..., p_n).
    4.  replace p_i by a shorter path p_i (a better response)
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Theorem 1 *Every traffic congestion game has a pure strategy N.E.*

Proof: Define a potential function and show each iteration improves the function in the finite space of pure strategies. For an edge $e \in E$ across which x cars are traveling in $(p_1, p_2, \ldots, p_n)$, define

$$Potential(e) = T_e(1) + T_e(2) + \ldots + T_e(e) = \sum_{j=1}^{x} T_e(j)$$

$$Potential(p_1, p_2, \ldots, p_n) = \sum_{e \in E} Potential(e)$$

In a typical iteration of the while-loop, let e be an arbitrary edge on path p_i. Let x be the number of cars traveling across e just before this iteration. $Potential(e)$ decreases by $T_e(x)$. That is, potential along p_i decreases by the sum of travel times along p_i. By the same argument, potential along e increases by $T_e(y)$ where y = volume of traffic along e after i joins this edge. In other words, potential along p_i increases by travel time along p_i Since travel time along p_i is strictly smaller than travel time along p_i, the potential strictly falls $\sum_{j=1}^{x} T_e(j)$ This shows that there always exists a pure strategy N.E. $\square$

Theorem 2 *For every traffic congestion game, there is a pure strategy N.E.(not every) whose cost is within 2 times of the cost of the welfare maximizing solution*

Proof: Let Z^* be a welfare maximizing strategy choice. In other words, Z^* minimizes the sum of travel times of all players. Let $TTT(Z)$ denote the sum of travel times of all players for a strategy choice Z. $xT_e(x)$ is the contribution of e to $TTT(Z)$.

$$Potential(e) = T_e(1) + T_e(2) + \ldots + T_e(x)$$

$$TTT(e) = T_e(x) + T_e(x) + \ldots + T_e(x) = xT_e(x)$$

Suppose $T_e(y) = a_e y + b_e$, the following figure shows the difference of the areas that represent $Potential(e)$ and $TTT(e)$.

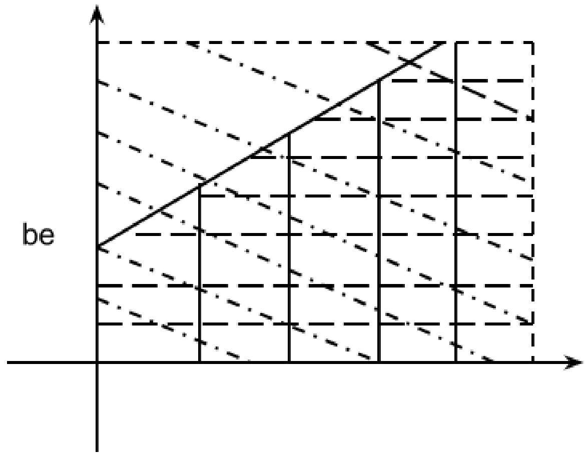

Figure 1: It is easy to see that $Potential(e) \geq TTT(e)/2$ in terms of the sum of the area.

We have shown $TTT(e)/2 \leq Potential(e) \leq TTT(e)$. For any strategy choice $Z : TTT(Z)/2 \leq Potential(Z) \leq TTT(Z)$ Suppose we run our algorithm starting from Z^* in the computation, $Z^* \longrightarrow Z$ (potential falls, TTT might increase)

$$\begin{aligned}
&\Rightarrow TTT(Z)/2 \leq Potential(Z^*) \leq TTT(Z^*) \\
&\Rightarrow TTT(Z)/2 \leq Potential(Z) \\
&\Rightarrow TTT(Z)/2 \leq TTT(Z^*) \\
&\Rightarrow TTT(Z) \leq 2TTT(Z^*)
\end{aligned}$$

$\square$

References

[1] Tim Roughgarden and Éva Tardos. How bad is selfish routing? *J. ACM*, 49(2):236–259, March 2002.