Lecture Notes: Social Networks: Models, Algorithms, and Applications
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1 Introduction to Game Theoretic Modelling of Traffic Congestion

Example 1: This is the first example of game on graph. This game is played by 4000 drivers. In the graph, A is the source and B is the target. The labels on the edges of the graph represent the delays on the route. The driver can choose any path among the two.

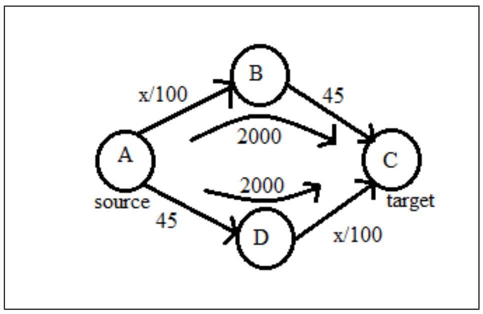

Figure 1: Diagram for the Example 1 showing the NE for the example in which 2000 drivers goes on both the paths

The travel time for each driver, when both routes have 2000 drivers will be: $20 + 45 = 65$ units of time.
Ques) Suppose we use the sum of travel time as the social welfare function than how good is the NE solution, relative to a choice that minimizes total travel time?
Suppose, p-drivers travel on path A $\rightarrow$ C $\rightarrow$ B
and the remaining 4000-p travels on A $\rightarrow$ D $\rightarrow$ B

$$\min_{0 \leq p \leq 4000} (\frac{p}{100} + 45)p + (4000 - p)(\frac{4000 - p}{100} + 45) \tag{1}$$

$$= \min_{0 \leq p \leq 4000} \frac{p^2}{100} + 45p + \frac{(4000 - p)^2}{100} + 4000 \times 45 \tag{2}$$

$$= \min_{0 \leq p \leq 4000} p^2 + (4000 - p)^2 \tag{3}$$

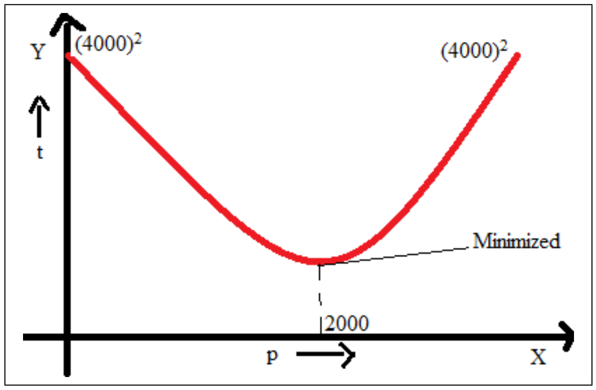

Figure 2: Graph showing that the incentive to deviate minimzes at 2000

NE is equal to welfare maxim choice.

Example 2: Figure: 1 refers to the problem in example 2, where there are have 4000 drivers with the choice of three paths.

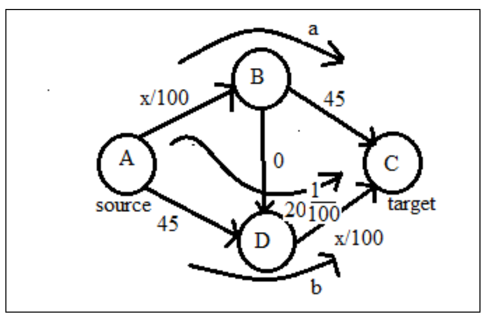

Figure 3: Diagram for the Example 2 showing the NE extra capacity

Ques) Is the NE from the previous example still a NE?
Answer) No, as a driver can do better by switching to A $\rightarrow C \rightarrow D \rightarrow$ B because the travel time on this path is much less than 65.

Ques) What is the NE solution?
Answer) Consider, a and b $>$ 0, a solution in which 'a' drivers take $A \rightarrow C \rightarrow B$, 'b' drivers take A $\rightarrow D \rightarrow B$ and 4000 - (a + b) take the path $A \rightarrow C \rightarrow D \rightarrow B$.

travel time on path $A \rightarrow C \rightarrow B : \frac{4000-b}{100} + 45 = 85 - \frac{b}{100}$

simmilarly, travel time on path $A \rightarrow D \rightarrow B : 85 - \frac{a}{100}$

and travel time on path $A \rightarrow C \rightarrow D \rightarrow B : 80 - \frac{a}{100} - \frac{b}{100}$
No, solution in which a driver uses $A \rightarrow C \rightarrow B$ or $A \rightarrow D \rightarrow B$ is NE. So, all 4000 drivers will travel on path, A $\rightarrow C \rightarrow D \rightarrow B$. This is the NE solution.

Observations:

1. Braess Paradox: Adding a capacity to the network led to a NE solution in which everyone is worse off.

2. The solution that maximizes social welfare is strictly better than the unique NE.

$\frac{cost\,of\,NE}{Cost\,of\,social\,welfare\,maximum\,solution}$ = Price of anarchy

In this example, price of anarchy $\geq \frac{80}{85}$

Results: (Roughgarden and tardos)

1. For any traffic network (G = (V, E), s,t), this game has a pure stratergy NE.

2. For any traffic network (G = (V,E), s,t), price of anarchy $\leq$ 2. [More complex algorithm bond it by $\frac{4}{3}$]

For linear delay functions, i.e.,

$$T_e(x) = a_e x + b_e$$

Consider the following "natural "algorithm:
Start with (p_1, p_2,...,p_n) is the choice of an st-path for each of the players 1,2,...,n.
while (p_1, p_2,...,p_n) is not a NE **do**
  Pick a player i who is not playing her best response and replace P_i by a path P_i' with strictly shorter travel time.
end while

Existence of a pure stratergy NE: Generally this type of algorithm is known as Tatomeat, and generally they don't converge but for the example 2, this algorithm converges when we find a path optimal for one p.