Lecture Notes: Social Networks: Models, Algorithms, and Applications
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1 Barabasi-Albert Preferential Attachment Model

Last time we discussed the Preferential Attachment Model which we will refer to as BA(M). Remember that there are a few key notions presented about this model:

- At $t = 0$, there is a single isolated node in the network called 0 (name refers to the time)
- At time t, node t arrives and connects to older nodes via m edges.For each new edge $t, j, 0 \leq j \leq t-1$, is picked with probability proportional to $deg_t(j)$ (i.e. degree of node j just before time step t)

Definition 1 ***Constant of Proportionality***

We define this property as follows:

$$\sum_j \cdot c \cdot deg_t(j) = 1 \Rightarrow c = \frac{1}{\sum_j \cdot deg_t(j)} = \frac{1}{2m(t)}$$

Taking the above definition into consideration, for our purposes of analysis set $m = 1$. In this case, just before time step t, there are $t-1$ edges in the graph. Then the value of c is simply:

$$c = \frac{1}{2(t-1)}$$

Continuing our train of thought, let $n_{k,t}$ = expected number of nodes with degree k just before time step t. Set up a recurrence for $n_{k,t+1}$ for $k > 1$.

$$n_{k,t+1} = n_{k,t} + n_{k-1,t} \cdot \frac{k-1}{2(t-1)} - n_{k,t} \cdot \frac{k}{2(t-1)}$$

$n_{k-1,t} \cdot \frac{k-1}{2(t-1)}$ represents the number of nodes entering which will be degree k while $n_{k,t} \cdot \frac{k}{2(t-1)}$ represents the number of nodes exiting due to being degree $k+1$.

For $k = 1$

$$n_{1,t+1} = n_{1,t} + 1 - n_{1,t+1} \cdot \frac{1}{2(t-1)}$$

Now let $p_{k,t}$ denote the expected fraction of nodes with degree k. Then $p_{k,t} = \frac{n_{k,t}}{t}$.

For $k > 1$

$$p_{k,t+1} \cdot (t+1) = p_{k,t} \cdot t + p_{k-1,t} \cdot t \cdot \frac{(k-1)}{2(t-1)} - p_{k,t} \cdot t \cdot \frac{k}{2(t-1)}$$

Now assume as $t \to \infty$, $p_{k,t}$ sequence converges. We will denote $\lim_{t\to\infty} p_{k,t} = p_k$

$$p_k \cdot (t+1) = p_k \cdot t + p_{k-1} \cdot \frac{(k-1)}{2} - p_k \cdot \frac{k}{2}$$

Simplifying the above gives us:

$$p_k = p_{k-1} \cdot \frac{(k-1)}{2} - p_k \frac{(k)}{2}$$

$$\Rightarrow p_k(\frac{2+k}{2}) = p_{k-1}(\frac{k-1}{2})$$

$$\Rightarrow p_k = p_{k-1}(\frac{k-2}{k+2})$$

$$\Rightarrow p_k = (\frac{k-1}{k+2}) \cdot (\frac{k-2}{k+1}) \cdot (\frac{k-3}{k})...\tfrac{1}{4} \cdot p_1 = \frac{6}{(k+2)(k+1)k} \cdot p_1$$

By using the same convergence assumption for the $k = 1$ recurrence, we get $p_1 = \frac{2}{3}$

Therefore $p_k = \dfrac{4}{k(k+1)(k+2)} \tilde{c} \cdot \dfrac{1}{k^3}$

2 Variant of Barabasi-Albert Model

The variant model has a few aspects that are different from the BA(m). When a new node arrives, it does (a) with probability p and (b) with probability $(1-p)$. Instead of using the other end point with a probability, this model does:

(a) Pick the other end point j of its edge with uniform probability

(b) Pick the other end pint j of its edge with probability proportional to $deg_t(j)$

Similar to the previous model, we can write the same type of recurrences.

For $k > 1$

$$n_{k,t+1} = n_{k,t} + nk - 1, t \cdot (\frac{p}{t} + (1-p)(\frac{k-1}{2(t-1)})) - n_{k,t} \cdot (\frac{p}{t} + \frac{(1-p)k}{2(t-1)})$$

Using fractions $p_{k,t}$ instead of expected sizes $n_{k,t}$ we get:

$$p_{k,t+1} \cdot (t+1) = p_{k,t} \cdot t + p_{k-1,t} \cdot t \cdot (\frac{p}{t} + \frac{(1-p(k-1))}{2(t-1)}) - p_{k,t} \cdot t \cdot (\frac{p}{t} + \frac{(1-p)k}{2(t-1)})$$

Taking limit as $t \to \infty$

$$p_k(t+1) = p_k \cdot t + p_{k-1}(p + \frac{(1-p)(k-1)}{2}) - p_k \cdot (p + \frac{(1-p)k}{2})$$

$$\Rightarrow p_k(1 + p + \frac{(1-p)k}{2}) = p_{k-1}(p + \frac{(1-p)(k-1)}{2})$$

$$\Rightarrow p_k(2(1+p) + (1-p)k) = p_{k-1}(2p + (1-p)(k-1))$$

$$\Rightarrow p_k = p_{k-1}(\frac{(1-p)(k-1) + 2p}{(1-p)k + 2(1+p)}) = p_{k-1}(\frac{(1-p)k + (3p-1)}{(1-p)k + 2p + 2})$$

$$= p_k = p_{k-1}(\frac{k + \frac{(3p-1)}{(1-p}}{k + \frac{2p+2}{(1-p)}})$$

The power law exponent is given by:

$$\frac{2p+2}{1-p} - \frac{(3p-1)}{(1-p)}$$
$$= \frac{(3-p)}{(1-p)}$$

Problem: Look at Easley-Kleinberg chp 18, Appendix for a different analysis:

$$\text{There the power law exponent} = 1 + \frac{1}{(1-p)}$$

There are many features of networks that are modelled that we have not considered:
–Community Structure
–Assortativity: Tendency of nodes of certain types to have more edges between them
More information these features can be found in Newman's paper.

References

[1] Barabasi, Albert. *The emergence of scaling in Random Networks.* Science 1999.