Solutions for Homework 1

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Question 1 Prove the *Tutte-Berge* Formula.

$$v(G) = min_{U \subseteq V} \frac{|V| + |U| - q(U)}{2}$$

Proof: For any subset $U \subseteq V$, we have

$$v(G) \leq |U| + v(G - U) \leq |U| + \frac{1}{2}(|V \setminus U| + q(U)) = \frac{|V| + |U| - q(U)}{2}$$

Hence, we are done if we show the reverse inequality. We can prove this by induction on $|V|$. First note that we can assume G is connected otherwise we can apply induction to each connected component of G. The case $|V| = \phi$ is trivial. If a graph G of order n is the smallest counter-example, there are two unmatched vertices $u, v \in G$. Select a matching M and the unmatched vertices u, v such that the distance between u and v is minimized. If $dist(u, v) = 1$, we can augment the matching M. Hence assume $dist(u, v) > 1$. There exists a matching N that misses a vertex t on the $u - v$ path. Select such a matching such that $|M \cap N|$ is maximized. By minimiality of $dist(u, v)$, we it follows that N covers u and v. Since both M and N are maximum size matching, there exists a vertex x covered by M but not by N. Let $x \in e = xy \in M$. Then y is covered by some edge $f \in N$. But $(N \setminus \{f\}) \cup \{e\}$ is a maximum matching that has a larger intersection with M, a contradiction.(Source : Schirjver's notes. http://homepages.cwi.nl/~lex/files/tutteb.pdf) □

Question 2 Let $G = (V, E)$ be a graph. An *edge cover* of $G = (V, E)$ is a set of edges $F \subseteq E$ such that for every vertex $v \in V$ there exists an edge in F incident on v. Let $\rho(G)$ denote the size of a smallest edge cover in G and let $\nu(G)$ denote the size of a largest matching in G. Prove that for any graph $G = (V, E)$ with no isolated vertices, $|V| = \nu(G) + \rho(G)$

Proof: Let M be a matching of size $\nu(G)$. For each of the $|V| - 2|M|$ vertices v missed by M, add to M an edge covering v. We obtain an edge cover of size $|V| - 2|M| + |M| = |V| - |M| = |V| - \nu(G)$. Hence, $\rho(G) \leq |V| - \nu(G)$. If F is an edge cover of size $\rho(G)$, for each $v \in V$ delete $d_F(v) - 1$ edges incident on v, where $d_F(v)$ is the degree of v in the graph induced by F. We obtain a matching of size at least $|F| - \sum_{v \in V}(d_F(v) - 1) = |F| - (2|F| - |V|) = |V| - |F| \leq |V| - \rho(G)$. Hence, $\nu(G) \geq |V| - \rho(G)$. (Source : http://homepages.cwi.nl/~lex/files/agtco.pdf) □

Question 3 Exercise 1, Chapter 2, page 40. To show that if a matching M of a bipartite graph G is suboptimal, there exists an M-augmenting path in G.

Proof: Let N be a matching of G of size larger than M. Let $H = (V, M \oplus N)$. It follows that each vertex of H has degree at most 2. Furthur, each component is either a path or a cycle. Since

$|N| > |M|$, there must be an odd path with more edges from N than M and this is an augmenting path of M. □

Question 4 Exercise 18, Chapter 3, page 64. Find a bipartite graph G with partition classes A and B such that for $H = G[A]$, there are at most $\frac{1}{2}\lambda_G(H)$ edge-disjoint H-paths in G.

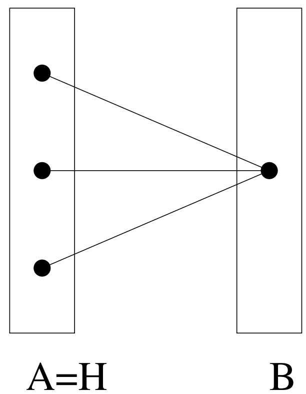

Figure 1: An example where there is only one edge-disjoint H-path, but $\lambda_G(H) = 2$

Question 5 Consider the following flow network with capacities $1, R$, and M. Assume that $M \geq 4$ is an integer and $R = (\sqrt{5} - 1)/2$. We will show that there is an infinite sequence of augmentations possible for this network.

1. Let $a_0 = 1$, $a_1 = R$, and $a_{n+2} = a_n - a_{n+1}$ for any $n \geq 0$. Show by induction that $a_n = R^n$.

   Proof: By induction. $a_2 = a_1 - a_0 = 1 - R = R^2$. Assume true for a_{n-1}, a_{n-2}, then $a_n = a_{n-2} - a_{n-1} = R^{n-2} - R^{n-1} = R^{n-2}(1 - R) = R^n$. □

2. Start with an initial flow f that assigns 1 unit of flow to edges $(s, c), (c, b)$ and (b, t) and 0 units everywhere else. Now notice that the residual capacities of edges (c, d) and (a, b), are a_0 and a_1 and the residual capacity of $(b, c) = 1$. Describe a sequence of 4 augmentations after which the residual capacities of edges $(c, d), (a, b)$ and (b, c) are $a_2 = 1 - R, a_3 = 2R - 1$ and 1 respectively.

   Solution : Four iterations :

   (a) Send R units of flow along the path $P_1 = sabcdt$. Now the residual capacities of the edges on P_1 are $(s, a) = M - R, (b, a) = R, (b, c) = 1 - R, (c, d) = 1 - R, (d, t) = M - R$.

   (b) Send R units of flow along the path $P_2 = scbat$. The residual capacities of the edges on P_2 are $(s, c) = M - 1 - R, (b, c) = 1, (a, b) = R, (a, t) = M - R$.

   (c) Send R^2 units of flow along the path $P_1 = sabcdt$. Now the residual capacities of the edges on P_1 are $(s, a) = M - (R + R^2), (a, b) = R^3, (b, c) = 1 - R^2, (c, d) = 1 - (R + R^2), (d, t) = M - (R + R^2)$.

   (d) Send R^2 units of flow along the path $P_3 = sdcbt$. The residual capacities of the edges along P_3 become $(s, d) = M - R^2, (c, d) = R^2, (b, c) = 1, (b, t) = M - 1 - R^2$.

3. Call the sequence of 4 augmentations described in (2) a *round*. Generalize your solution to (2) and show a sequence of of n rounds after which the residual capacities of the edges $(c, d), (a, b)$ and (b, c) are respectively a_{2n}, a_{2n+1}, and 1. What is the value of the flow at this point ? What is the limiting value of the flow as $n \to \infty$?

Solution :

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For i = 1 to n do
    1.  send R^{2i-1} = a_{2i-1} units of flow through P_1
    2.  send R^{2i-1} = a_{2i-1} units of flow through P_2
    3.  send R^{2i} = a_{2i} units of flow through P_1
    4.  send R^{2i} = a_{2i} units of flow through P_3
    5.  i <- i + 1
End For
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From the solution to the previous question we know that the residual capacities of edges $((c, d), (a, b), (b, c))$ are respectively $(a_2, a_3, 1)$ at the end of round 1. Assume that the claim holds for rounds $i \leq n$. Hence, the residual capacities of $((c, d), (a, b), (b, c))$ are respectively $(a_{2n}, a_{2n+1}, 1)$. Consider the execution of round $n + 1$.

- Send a_{2n+1} units of flow through P_1. The residual capacities become $(a_{2n} - a_{2n+1}, a_{2n+1} - a_{2n+1}, 1 - a_{2n+1})$.
- Send a_{2n+1} units of flow through P_3. The residual capacities become $(a_{2n+2}, a_{2n+1}, 1)$.
- Send a_{2n+2} units of flow through P_1. The residual capacities become $(a_{2n+2} - a_{2n+2}, a_{2n+1} - a_{2n+2}, 1 - a_{2n+2})$.
- Send a_{2n+2} units of flow through P_3. The residual capacities become $(a_{2n+2}, a_{2n+3}, 1)$.

as required.

The value of the flow after n rounds is

$$
\begin{aligned}
1 + 2\sum_{i=1}^{n}(a_{2n-1} + a_{2n}) &= 1 + 2\sum_{i=1}^{2n} a_i \\
&= 1 + 2a_1 + 2[(a_0 - a_1) + (a_1 - a_2) + \cdots + (a_{2n-2} - a_{2n-1})] \\
&= 1 + 2R + 2(a_0 - a_{2n-1}) \\
&= 3 + 2R - 2a_{2n-1}
\end{aligned}
$$

The limiting value of this flow as $n \to \infty$ is $3 + 2R = 3 + (\sqrt{5} - 1) \approx 4.23$ However, the optimum flow in the network is clearly $2M + 1$, and for $M \geq 4$, the optimum value of the flow is atleast 9. Hence, the Ford-Fulkerson algorithm does not converge to an optimum flow. (Read More about it in : http://www.cs.tau.ac.il/~zwick/papers/flow.ps.gz)

Question 6 Suppose G is an r-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that G has a 1-factor.

Proof: We will show that Tutte's condition holds and hence G has a 1-factor. Tutte's condition states that $\forall S \subseteq V$, if $|S| \geq q(G-S)$ holds, then G has a 1-factor. Here $q(G-S)$ is the number of odd components in $G-S$. Since G is r-connected, we only need to consider subsets S where $|S| \geq r$, otherwise G remains connected. For a given subset S, let $C_1, \cdots, C_m$ be the odd components of $G-S$. Since G is r-connected it follows that there are at least r edges from each C_i to *distinct* vertices of S. Hence there are at least mr edges crossing S. If $m > |S|$ there are $|S|$ vertices which have at least $mr > r|S|$ edges adjacent to them. Hence, must be at least one vertex with $r+1$ edges from distinct odd components which yields an induced $K_{1,r+1}$. Hence, $m \leq |S|$ and Tutte's condition holds. $\square$

Question 7 Let $A = (A_1, \cdots, A_m)$ be a collection of subsets of a set Y. A *system of distinct representatives* (SDR) for A is a set of distinct elements $a_1, \cdots, a_m$ in Y such that $a_i \in A_i$. Prove that A has an SDR iff $|\cup_{i\in S} A_i| \geq |S|$ for all $S \subseteq \{1, \cdots, m\}$.
Proof: Consider the following bipartite graph $H = (A, Y, E)$ where $(A_i, x) \in E$ iff $x \in A_i$. Now the result follows from application of Hall's theorem on this graph. $\square$

Question 8 Let $N = (G, s, t, c)$ be a flow-network and suppose $(S, \overline{S})$ and $(T, \overline{T})$ are two minimum capacity cuts of N. Recall that if $(A, \overline{A})$ is a cut of N, then $s \in A$ and $t \in \overline{S}$. Prove that $(S \cup T, \overline{S \cup T})$ and $(S \cap T, \overline{S \cap T})$ are also a minimum cuts of N.
Proof: If $S \subseteq T$ or $T \subseteq S$, the result is trivial. Hence, assume this is not the case. Let us define the following sets.

$$\begin{aligned}
X &= \{(x,y) \mid (x,y) \in (S,\overline{S}) \cap (T,\overline{T})\} \\
A &= \{(x,y) \mid x \in S, y \in \overline{S \cup T}\} - X \\
B &= \{(x,y) \mid x \in T, y \in \overline{S \cup T}\} - X \\
C &= \{(x,y) \mid x \in S, y \in \overline{S} \cap T\} \\
D &= \{(x,y) \mid x \in T, y \in S \cap \overline{T}\}
\end{aligned}$$

Now, we can express $(S, \overline{S})$ and $(T, \overline{T})$ in terms of these sets.

$$(S, \overline{S}) = A + C + X$$
$$(T, \overline{T}) = B + D + X$$

Hence,

$$\begin{aligned}
(S,\overline{S}) + (T,\overline{T}) &= A + B + C + D + 2 \cdot X \\
&= (A + B + X) + (C + D + X) \\
&= (S \cup T, \overline{S \cup T}) + (S \cap T, \overline{S \cap T})
\end{aligned}$$

Since $(S, \overline{S})$ and $(T, \overline{T})$ are min-cuts of the graph, equality is achieved only when the values of both cuts on the RHS of the equation above are equal to the min-cost cut. $\square$