22C:137/22M:152 Homework 1

Due: Tuesday, 3/1

Notes: (a) Solve all 8 problems listed below. We will grade some subset of 5 problems. (b) It is possible that solutions to some of these problems are available to you via other graph theory books or on-line lecture notes, etc. If you use any such sources, please acknowledge these in your homework. You will benefit most from the homework, if you sincerely attempt each problem on your own first, before seeking other sources. (c) It is okay to discuss these problems with your classmates. Just make sure that you take no written material away from these discussions.

1. Prove the following *Tutte-Berge* formula:
$$\upsilon(G) = \min_{U \subseteq V} \frac{|V| + |V - U| - q(U)}{2},$$
where $G = (V, E)$ is graph, $\upsilon(G)$ is the size of a largest matching in G, and $q(U)$ is the number of odd components in the subgraph of G induced by U.
Hint: Use Tutte's 1-factor theorem to prove this.

2. Let $G = (V, E)$ be a graph. An *edge cover* of $G = (V, E)$ is a set of edges $F \subseteq E$ such that for every vertex $v \in V$ there exists an edge in F incident on v. Let $\rho(G)$ denote the size of a smallest edge cover in G and let $\upsilon(G)$ be the size of a largest matching in G. Prove that for any graph $G = (V, E)$ with no isolated vertices, $|V| = \upsilon(G) + \rho(G)$.
Note: This is usually called *Gallai's Theorem.*

3. Exercise 1, Chapter 2, page 40.

4. Exercise 18, Chapter 3, page 64.

5. Consider the following flow network (see Figure 1) with edge capacities 1, R, and M. Assume that $M \geq 4$ is an integer and $R = (\sqrt{5} - 1)/2$. We will show that there is an infinite sequence of augmentations possible for this flow network.
   (a) Let $a_0 = 1$, $a_1 = R$, and $a_{n+2} = a_n - a_{n+1}$ for any $n \geq 0$. Show by induction that $a_n = R^n$.
   (b) Start with an initial flow f that assigns 1 unit of flow to edges (s, c), (c, b), and (b, t) and 0 units everywhere else. Now notice that the residual capacities of edges (c, d) and (a, b), are a_0 and a_1 and the residual capacity of (b, c) is 1. Describe a sequence of 4 augmentations after which the residual capacities of edges (c, d), (a, b), and (b, c) are $a_2 = 1 - R$, $a_3 = 2R - 1$, and 1.
   (c) Call the sequence of 4 augmentations described in (b) a *round.* Generalize your solution to (b) and show a sequence of n rounds after which the residual capacities of edges (c, d), (a, b), and (b, c) are respectively a_{2n}, a_{2n+1}, and 1. What is the value of the flow at this point? What is the limiting value of the flow as $n \to \infty$.

6. Suppose that G is an r-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that G has a 1-factor.

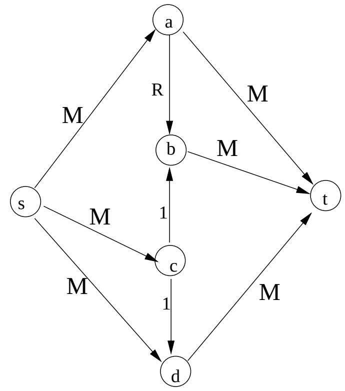

Figure 1: A flow-network on which the Ford-Fulkerson method does not terminate.

7. Let $\mathbf{A} = (A_1, A_2, \ldots, A_m)$ be a collection of subsets of a set Y. A *system of distinct representatives* (SDR) for $\mathbf{A}$ is a set of distinct elements $a_1, a_2, \ldots, a_m$ in Y such that $a_i\ inA_i$. Prove that $\mathbf{A}$ has an SDR iff $|\cup_{i \in S} A_i| \geq |S|$ for every $S \subseteq \{1, 2, \ldots, m\}$.

8. Let $N = (G, s, t, c)$ be a flow network and suppose that $(S, \overline{S})$ and $(T, \overline{T})$ are two minimum capacity cuts of N. Recall that if $(A, \overline{A})$ is a cut of N, then $s \in A$ and $t \in \overline{A}$. Prove that $(S \cup T, \overline{S \cup T})$ and $(S \cap T, \overline{S \cap T})$ are also minimum cuts in N.
