22C:137/22M:152 Final Exam

Due: Thursday, 5/12, 5 pm

Notes: (a) Solve all 4 problems listed below. (b) You are not to discuss these problems with your classmates or anyone else, except for your TA and the instructor. You are also not allowed to use any sources other than the textbook, Schrijver's notes, the GLS book, and your notes from my lectures. (c) You are welcome to see me during my office hours 2-3 on Friday (5/6), Monday (5/9) or Wednesday (5/11), or set up an alternate time to meet, or ask questions by e-mail. (d) Each problem is worth 50 points. (e) You can turn in your exam by e-mail, in my mailbox, or to me in person by 5 pm on Thursday, 5/12.

1. For each of the classes of graphs defined below, determine if the class of graphs is perfect or not. Prove your answer. We say that a class of graphs is perfect, if every graph in the class is a perfect graph.

   (i) **Mystery graph class 1.** Let π be a permutation of the numbers in $\{1, 2, \dots, n\}$. The graph $G(\pi)$ has vertex set $\{1, 2, \dots, n\}$ and edges $\{i, j\}$ iff π switches the order of i and j. In other words, for any $i < j$, $\{i, j\}$ is an edge iff i appears after j in π. The mystery graph class 1 consists of graphs $G(\pi)$ for all permutations π of $\{1, 2, \dots, n\}$ for all integers $n \geq 1$.

   (ii) **Mystery graph class 2.** Let S be a finite set of chords of a circle. The graph $G(S)$ has vertex set S and edges $\{c, c'\}$ iff the chords c and c' intersect. It does not really matter, but just to be concrete, you may think of the chords as *closed* line segments. The mystery graph class 2 consists of graphs $G(S)$ for all finite sets of chords in a circle.

2. Show that almost every graph $G \in \mathcal{G}(n, 1/2)$ has at least $n^{1/3}$ vertices of degree precisely $\lfloor n/2 \rfloor$.
   (**Hint:** Compute the expectation and the variance of the number of these vertices.)

3. Prior to the simple construction of triangle-free graphs with arbitrary chromatic number proposed by Mycielski, Blanches Descartes (a pseudonym used by William Tutte) proposed a more complicated construction of graphs with girth at least six and arbitrary chromatic number. Here it is.

   Let G be an n-vertex graph with girth at least six and chromatic number $k \geq 2$. Form a new graph H as follows. Take $\binom{nk}{n}$ disjoint copies of G and a set S of kn vertices, and set up a one-one correspondence between the copies of G and the n-element subsets of S. For each copy of G, join its vertices to the members of the corresponding n-element subset of S by a matching.

   Show that H has chromatic number at least $k + 1$ and girth at least six.

4. Let G be a graph with vertex set $\{1, 2, \ldots, n\}$. Clearly, every vector $x \in \mathrm{STAB}(G)$ satisfies

$$\begin{aligned} x_i &\geq 0 \text{ for all } i \in V(G) && (1) \\ x_i + x_j &\leq 1 \text{ for all } \{i, j\} \in E(G) && (2) \end{aligned}$$

Therefore if we let $\mathrm{ESTAB}(G)$ denote the set of all vectors $x \in \mathbf{R}^n$ satisfying (1) and (2), then $\mathrm{STAB}(G) \subseteq \mathrm{ESTAB}(G)$. Also, note that $\mathrm{QSTAB}(G) \subseteq \mathrm{ESTAB}(G)$.

Show that $\mathrm{STAB}(G) = \mathrm{ESTAB}(G)$ iff G is a bipartite graph with no isolated nodes.
