22C:137/22M:152 Midterm Exam Solutions

Notes: (a) Solve all 4 problems listed below. (b) You are not to discuss these problems with your classmates or anyone else. You are also not allowed to use any sources other than the textbook, Schrijver's notes, and your notes from my lectures. (c) You are welcome to see me during my office hours 2-3 on Wednesday, or set up an alternate time to meet, or ask questions by e-mail. (d) Each problem is worth 50 points.

1. Suppose $G = (X, Y, E)$ is a bipartite graph. Let H be the graph obtained from G by adding one vertex to Y if $|G|$ is odd and then adding the edges of a clique on the vertices in Y.

   (i) Prove that G has a matching of size $|X|$ iff H has a 1-factor.

   (ii) Prove that if G satisfies Hall's condition, that is, $|N(S)| \geq |S|$ for all $S \subseteq X$, then H satisfies Tutte's condition, which is that, $q(H - T) \leq |T|$ for all $T \subseteq V(H)$.

   (iii) Use items (i) and (ii) to derive Hall's theorem from Tutte's theorem.

   Solution to 1(i): If G has a matching of size X and let M be such a matching. Then $|X|$ vertices in Y are matched by M, leaving $|Y| - |X|$ vertices in Y unmatched. Since $|H|$ is even, $|Y| - |X|$ is even and since $H[Y]$ is a clique we can pick a matching M' of vertices in Y not matched by M. $M \cup M'$ is a 1-factor of H.

   If H has a 1-factor, say M then the maximal subset $M_X \subseteq M$ of edges incident on X is a matching of X in G.

   Solution to 1(ii): Suppose H violates Tutte's condition, i.e., $q(H - T) > |T|$ for some $T \subseteq V(H)$. Since $Y \subseteq V(H)$ is a clique, one of the components C in $H - T$ contains all of $Y - T$ and the remanining components are singletons from X. Let k be the number of singletons in $H - T$. Then, $q(H - T) = k + 1$, if $|C|$ is odd, and $q(H - T) = k$, otherwise. If $|C|$ is even, then $q(H - T) \leq |T|$ from Hall's theorem.

   On the other hand, if $|C|$ is odd, then $V(H) = |T| + k + |C|$. Since $|V(H)|$ is even and $|C|$ is odd, $|T| + k$ must be odd. Since $q(H - T) = k + 1 > |T|, k \geq |T|$. The fact that $|T| + k$ is odd rules out $k = |T| \Rightarrow k > |T|$. Let S be this set of k vertices. Then $N(S) \subseteq T \cap Y$ and therefore $|N(S)| \leq |T \cap Y| \leq |T| < k = |S|$ violating Hall's condition.

   Solution to 1(iii): Suppose $\forall\ S \subseteq X$, $|N(S)| \geq |S|$. Then, by (iii) H satisfies Tutte's condition and therefore has a 1-factor. By (i) this implies that G has a matching of X.

2. (i) Prove that $\kappa'(G) = \kappa(G)$ if G is a 3-regular simple graph. (ii) Find with proof the smallest 3-regular graph with connectivity 1. (iii) Use this to obtain a simple proof that the Petersen graph is 3-connected.

   Solution to 2(i) : $\kappa(G) \leq \kappa'(G)$ always. We now show that $\kappa(G) \geq \kappa'(G)$. Let $t = \kappa'(G)$. Then for any pair of vertices, $u, v \in V(G)$, $\{u, v\} \notin E(G)$, there are t edge disjoint paths between u and v. If two of these paths share a vertex w, $w \notin \{u, v\}$, then $degree(w) \geq 4$ which is impossible since G is 3-regular. Therefore, there are t internally vertex disjoint paths between u and v. Since the choice of u and v is arbitrary, it follows that $\kappa(G) \geq t$.

Solution to 2(ii) : If $\kappa(G) = 1$, then by (i), $\kappa'(G) = 1$. Therefore, G has a bridge. Let e be a bridge in G. $G - e$ has two connected components, call these H_1 and H_2. H_1 has one vertex with degree 2 and the rest of degree 3. Since H_1 has one degree 2 vertex, it has at least three vertices. Hence it has at least one degree 3 vertex. This means that it has at least 4 vertices. Can H_1 have one degree 2 vertex and 3 degree 3 vertices ? No, because a graph has an even number of odd degree vertices. Hence, $|H_1| \geq 5$. Similarly, $|H_2| \geq 5$ and therefore $|G| \geq 10$. One example of a 3-regular 10-vertex graph G with $\kappa(G) = 1$ is shown in Figure 1.

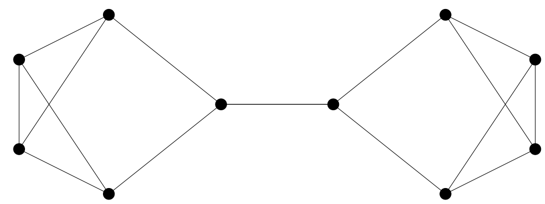

Figure 1: An example of the smallest 3-regular graph which is 1-connected

Solution to 2(iii) : Consider the labelling of the Petersen graph as shown in Figure 2. Let $A = \{a_1, \cdots, a_5\}$ and $B = \{b_1, \cdots, b_5\}$. Note that $G[A]$ and $G[B]$ are both 5-cycles. For any vertices a_i, a_j, $i \neq j$, there are two edge-disjoint $a_i - a_j$ paths in $G[A]$ because $G[A]$ is a cycle. Also, let p_{ij} be a path between b_i and b_j in $G[B]$. Then $a_i - p_{ij} - a_j$ is a path that is edge disjoint from any path in $G[A]$. Hence, there are at least 3 edge-disjoint $a_i - a_j$ paths in G. The same argument holds for vertices b_i and b_j, $i \neq j$.

Now consider a pair a_i, b_j. Let p^A_{ij} be a shortest $a_i - a_j$ path in $G[A]$ and let p^B_{ij} be a shortest $b_i - b_j$ path in $G[B]$. Since $G[A]$ and $G[B]$ are 5-cycles, $|p^A_{ij}| \leq 2$ and $|p^B_{ij}| \leq 2$. Therefore, there exists $k \in \{1, \cdots, 5\}$ $k \neq i$ and $k \neq j$ such that p^A_{ij} is not incident on a_k and p^B_{ij} is not incident on b_k. Let q^A be an $a_i - a_k$ path in $G[A]$ that is edge-disjoint from p^A_{ij} and let q^B be a $b_k - b_j$ path in $G[B]$ edge-disjoint from p^B_{ij}. Then $a_i - b_i - p^B_{ij}$, $a_i - p^A_{ij} - b_j$ and $a_i - q^A - a_k - b_k - q^B$ are 3 edge-disjoint $a_i - b_j$ paths. Since $\kappa'(G) = 3 \Rightarrow \kappa(G) = 3$.

3. Let X be a finite set and let $r : 2^X \to \mathbb{Z}$.
   (i) Show that if $r = r_M$ for some matroid M on X then r satisfies the following conditions:
      (a) $0 \leq r(Y) \leq |Y|$ for each subset Y of X;
      (b) $r(Z) \leq r(Y)$ whenever $Z \subseteq Y \subseteq X$;
      (c) $r(Y \cap Z) + r(Y \cup Z) \leq r(Y) + r(Z)$ for all $Y, Z \subseteq X$.
   (ii) Now show the converse: Suppose that $r : 2^X \to \mathbb{Z}$ satisfies conditions (a)-(c). Let $\mathcal{I} = \{Y \subseteq X \mid r(Y) = |Y|\}$. Show that $M = (X, \mathcal{I})$ is a matroid.

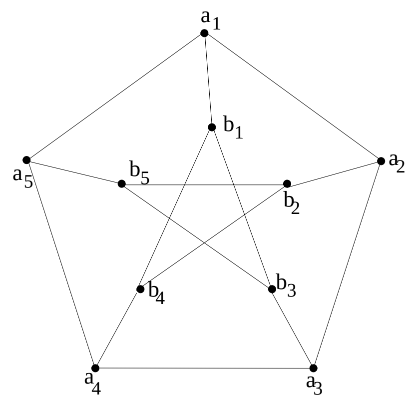

Figure 2: The petersen graph and a labelling

Solution to 3(i): (a) For any $Y \subseteq X$, $r_M(Y)$ is the common size of an inclusion-wise maximal independent subset of Y. Since ϕ is an independent set, $r_M(Y)$ is always defined and $r_M(Y) \geq 0$. The size of any subset of Y is at most Y and therefore $r_M(Y) \leq |Y|$.
(b) Let B be a basis of Z. Then $r_M(Z) = |B|$. Since $Z \subseteq Y$, it follows that $B \subseteq Y$. Furthermore, since B is independent $r_M(Y) \geq |B| = r_M(Z)$.
(c) Let B_1 be a basis of $Y \cap Z$. Let B_2 be a basis of $Y \cup Z$ that contains B_1. Note that $B_2 \cap (Y \cap Z) = B_1$, otherwise $Y \cap Z$ would contain an independent set larger than B_1 — which is not possible. Therefore, B_2 can be partitioned into B_Y, B_Z, and B_1 where $B_Y \subseteq Y - Z$ and $B_Z \subseteq Z - Y$. Therefore, $r(Y \cap Z) + r(Y \cup Z) = |B_Y| + |B_Z| + 2|B_1|$. Since $B_Y \cup B_1 \subseteq B_2$, $B_Y \cup B_1$ is an indendent set (contained in Y) and therefore $r(Y) \geq |B_Y| + |B_1|$. Similarly, $r(Z) \geq |B_Z| + |B_1|$. Therefore, $r(Y) + r(Z) \geq |B_Y| + |B_Z| + 2|B_1|$.

Solution to 3(ii): We show that $M = (X, \mathcal{I})$ is a matroid by showing that the three axioms of a matroid are satisfied.

Axiom 1. From (a) it follows that $0 \leq r(\phi) \leq |\phi| = 0$ and therefore $\phi \in \mathcal{I}$.

Axiom 2. Let $Y \subseteq X$ be an independent set and let $Z \subseteq Y$ be arbitrary. From (b) it follows that $r(Z) \leq |Z|$. If $r(Z) = |Z|$ then $Z \in \mathcal{I}$ and we are done. So we assume that $r(Z) < |Z|$. Then

$$r((Y - Z) \cap Z) + r((Y - Z) \cup Z) = r(\phi) + r(Y) = |Y|.$$

Also,

$$r(Y - Z) + r(Z) < |Y - Z| + |Z| = |Y|.$$

From (c) we have that

$$r((Y - Z) \cap Z) + r((Y - Z) \cup Z) \leq r(Y - Z) + r(Z),$$

but this implies that $|Y| < |Y|$, a contradiction.

Axiom 3. Let $Y, Z \in \mathcal{I}$ such that $|Y| < |Z|$. If there is an $x \in Z - Y$ such that $Y \cup \{x\} \in \mathcal{I}$, we are done. So we assume that for all $x \in Z - Y$, $Y \cup \{x\}$ is not an independent set. (a) implies that $r(Y \cup \{x\}) \leq |Y| + 1$. Since $Y \cup \{x\}$ is not an independent set, $r(Y \cup \{x\}) \leq |Y|$. Furthermore, (b) implies that $r(Y \cup \{x\}) \geq |Y|$ and hence $r(Y \cup \{x\}) = |Y|$. Now consider any subset $S \subseteq Z - X$. We will show by induction on $|S|$ that $r(Y \cup S) = |Y|$. We have already shown this for all $S \subseteq Z - X$ with $|S| = 1$. So suppose that $S \subseteq Z - X$ with $|S| \geq 2$ and let x and y be distinct elements in S. Let $A = Y \cup (S - \{x\})$ and $B = Y \cup (S - \{y\})$. Then,

$$r(A \cup B) + r(A \cap B) = r(Y \cup S) + r(Y \cup (S - \{x, y\}) \leq r(Y \cup S) + |Y|.$$

The last inequality follows from the induction hypothesis. Also,

$$r(A) + r(B) = r(Y \cup (S - \{x\})) + r(Y \cup (S - \{y\})) = 2|Y|.$$

The last equality follows from the induction hypothesis. Using (c) we get that $r(Y \cup S) \leq |Y|$. Using (b) we get that $r(Y \cup S) \geq |Y|$, which implies that $r(Y \cup S) = |Y|$.

Now let $S = Z - Y$. Then $r(Y \cup S) = |Y|$ by the above argument. Also, $r(Y \cup S) = r(Z) = |Z|$ since Z is an independent set. However, $|Y| < |Z|$ and so we have a contradiction.
