Assignment 1 (Mathematical background, 200 points)

This assignment asks you to solve the following 12 problems (most of them taken from the textbook). The goal of this assignment is to refresh student's ability to handle mathematical notations, definitions, theorems, and proofs.

1. (12 points) Write formal descriptions of the following sets:

   (a) The set containing the numbers 1, 10, 100 (3 points)

   Solution: $\{1, 10, 100\}$

   (b) The set containing all integers greater than 5 (3 points)

   Solution: $\{n|n > 5\}$

   (c) The set containing the empty string (3 points)

   Solution: $\{\epsilon\}$

   (d) The set containing nothing at all (3 points)

   Solution: $\emptyset$

2. (15 points) One claim of set theory is that every mathematical object can be expressed as a set. You are required to show that this is true by solving the problems:

   (a) Shows that the definition of a pair $(x, y) = \{x, \{x, y\}\}$ is correct (5 points).

   Solution sketch: (1) Show first that the definition preserve the tuples equality, i.e., if $(x, y) = (u, v)$ then their set representations are the same. (2) Show that if the set represented by (x, y) is equal to the set represented by (u, v) then the tuples (x, y) and (u, v) are equal.

   (b) Construct the set representing the triple (x, y, z) (5 points).

   Solution sketch: Use the equality: $(x, y, z) = ((x, y), z) = \{(x, y), \{(x, y), z\}\} = \ldots$

   (c) Construct the set representation of a list $L = (x_1, x_2, \ldots, x_n)$ and show that your construction is correct (5 points).

   Solution sketch: Use induction.

3. (5 points) Let $A = \{x, y, z\}$ and $B = \{x, y\}$. Answer each of the following questions:

   (a) Is A a subset of B? Justify your answer (1 points).

   (b) Construct the set $A \cup B$ (1 points).

   (c) Construct the set $A \cap B$ (1 points).

   (d) Construct the set $A \times B$ (1 point).

   (e) Construct the power set of A (1 point).

Solution sketch: obvious.

4. (10 points) If C is a set with c elements how many elements are in the power set of C (5 points). Justify your answer (5 points)

   Solution: $|\mathcal{P}(C)| = 2^c$.
   Reason: $\mathcal{P}(C)$ is the set of subsets of C. For each $c \subseteq C$, $c \in \mathcal{P}(C)$ or $c \notin \mathcal{P}(C)$. Hence, each element of C doubles the number of subsets of C. Alternatively, we can view each subsets $S \subseteq C$ as corresponding to a binary string b of length c where S contains the i-th element of C iff i-th place of b is 1. There are 2^c strings of length c and hence that many subsets of C.

5. (5 points) If a set A has a elements and a set B has b elements, how many elements has the set $A \times B$? **Solution sketch:** ab.

6. (28 points) For each part, give a relation that satisfies the condition and show that the condition is satisfied.

   (a) Reflexive and symmetric but not transitive (7 points).
   (b) Reflexive and transitive but not symmetric (7 points).
   (c) Symmetric and transitive but not reflexive (14 point).

   Sketch solution: Consider the set $\mathcal{N}$ of natural numbers as the underlying set. Then we have:

   (a) $R = \{(a, b)| \ |a - b| \leq 1\}$
   (b) $R = \{(a, b)|a \leq b\}$
   (c) $R = \{(a, b)|a \neq 1 \wedge b \neq 1\}$

7. (5 points). Consider the undirected graph $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 3), (1, 3), (2, 4), (1, 4)\}$. Draw the graph G (1 point). Attach to each node on the graph G its degree (4 points).

   Solution sketch: obvious.

8. (10 points) Find the error in the following proof that $2 = 1$.

   Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$; subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$; factor each side to get $(a+b)(a-b) = b(a-b)$; divide by $(a - b)$ to get $a + b = b$; let $a = b = 1$; the result is $2 = 1$

   Solution: it is illegal to divide by zero!

9. (20 points) Find the error in the following proof that all horses are the same color.

   Claim: In a set of n horses, all are of the same color.
   Proof: by induction on n.

- **Basis:** For $n = 1$ the set contains just one horse and hence all horses are of the same color.
- **Induction step:** For $n > 1$ assume that the claim is true for $n = k$ and prove that it is true for $n = k+1$. Take any set set H of $k+1$ horses. We show that all the horses in this set have the same color. For that we remove one horse, h_1 from H thus obtaining the set H_1 of k horses, which have the same color. Now, bring back in H_1 the horse h_1 and remove another horse from H_1 thus obtained, different from h_1, thus obtaining the set H_2. By the same argument all horses in H_2 have the same color. Hence, all horses in H must have the same color, what completes the proof.

Solution sketch: If H contains at least 3 horses, then H_1 and H_2 contain a horse in common, so the argument works properly. But if H contains exactly 2 horses, then H_1 and H_2 contains exactly 1 horse but do not have a horse in common. Hence we cannot conclude that the horses in H_1 have the same color with the horses in H_2. Hence, the two horses in H may not be colored the same.

10. (30 points) Show that every simple graph (for each node $n \in G$, no edge (n, n) allowed) with 2 or more nodes contains two nodes that have equal degrees.

    Solution: Let G be a graph with n nodes, $n \geq 2$. The degree of every node in G is one of the n possible values 0 to $n-1$. We would like to use the pigeon hole principle to show that two of these values must be the same, but number of possible values is too big. However, not all of these values can occur in the same graph because a node of degree 0 cannot coexist with a node of degree $n-1$. Hence G can exhibit at most $n-1$ degree values among its n nodes. So two of the values must be the same.

11. (40 points) **Ramsey's theorem:** Let G be a graph. A *clique* in G is a subgraph of G in which every two nodes are connected by an edge. An *anti-clique*, also called an *independent set*, is a subgraph of G in which every two nodes are not connected by an edge. Show that every graph with n nodes contains either a clique or an anti-clique with at least $\frac{1}{2}log_2 n$ nodes (40 points).

    Proof: split the nodes of G in two piles, A and B by the procedure:

    $A := \emptyset$, $B := \emptyset$; $G_{rest} = nodes(G)$
    while $(G_{rest} \neq \emptyset)$ take a node $x \in G_{rest}$
        If $degree(x) > |G|/2$ then add x to A, i.e., $A := A \cup \{x\}$
        remove from G_{rest} all nodes to which x is not connected;

        If $degree(x) \leq |G|/2$ then add x to B, i.e., $B := B \cup \{x\}$
        removes from G_{rest} all nodes to which x is connected.

    Note: at most half of the nodes of G are removed from G_{rest} at each step of the above procedure. Hence, at least $log_2 n$ steps will occur before procedure terminates.

Each step adds a node to one of the piles A or B, so one of the pile ends up with at least $\frac{1}{2}log_2 n$ nodes. A pile contains the nodes of a clique and B pile contains the nodes of an anti-clique.

12. **Theorem:** for each $t \geq 0$, let P_t be the amount of the loan outstanding after t month, Y be the monthly payment, I be the yearly interest rate of the loan and $M = 1 + I/12$ be the monthly multiplier. Then P_t is computed by the formula:

$$P_t = PM^t - Y\frac{M^t - 1}{M - 1}$$

(20 points) You are required to derive a formula for calculating the size of the monthly payment for a mortgage in term of the principal P, interest I, and the number of payments t. Assume that, after t payments have been made, the loan amount is reduced to 0. Use the formula to calculate the dollar amount of each monthly payment for a 30-year mortgage with 360 monthly payments on an initial loan amount of \$100,000 with 5% annual interest rate.

Solution: Let $P_t = 0$ and solve the equation in the theorem for Y to get $Y = PM^t(M-1)/(M^t-1)$. For $P = 100,000$, $I = 0.05$, and $t = 360$ we have $M = 1+(0.05)/12$. Using a calculator one gets that $Y \approx \$536.82$ is the monthly payment.

Due date is Wednesday 16 September 2009.