Assignment 1 (Mathematical background, 200 points)

This assignment asks you to solve the following 12 problems (most of them taken from the textbook). The goal of this assignment is to refresh student's ability to handle mathematical notations, definitions, theorems, and proofs.

1. (12 points) Write formal descriptions of the following sets:
   (a) The set containing the numbers 1, 10, 100 (3 points)
   (b) The set containing all integers greater than 5 (3 points)
   (c) The set containing the empty string (3 points)
   (d) The set containing nothing at all (3 points)

2. (15 points) One claim of set theory is that every mathematical object can be expressed as a set. You are required to show that this is true by solving the problems:
   (a) Shows that the definition of a pair $(x, y) = \{x, \{x, y\}\}$ is correct (5 points).
   (b) Construct the set representing the triple (x, y, z) (5 points).
   (c) Construct the set representation of a list $L = (x_1, x_2, \ldots, x_n)$ and show that your construction is correct (5 points).

3. (5 points) Let $A = \{x, y, z\}$ and $B = \{x, y\}$. Answer each of the following questions:
   (a) Is A a subset of B? Justify your answer (1 points).
   (b) Construct the set $A \cup B$ (1 points).
   (c) Construct the set $A \cap B$ (1 points).
   (d) Construct the set $A \times B$ (1 point).
   (e) Construct the power set of A (1 point).

4. (10 points) If C is a set with c elements how many elements are in the power set of C (5 points). Justify your answer (5 points)

5. (5 points) If a set A has a elements and a set B has b elements, how many elements has the set $A \times B$?

6. (28 points) For each part, give a relation that satisfies the condition and show that the condition is satisfied.
   (a) Reflexive and symmetric but not transitive (7 points).
   (b) Reflexive and transitive but not symmetric (7 points).
   (c) Symmetric and transitive but not reflexive (14 point).

7. (5 points). Consider the undirected graph $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 3), (1, 3), (2, 4), (1, 4)\}$. Draw the graph G (1 point). Attach to each node on the graph G its degree (4 points).

8. (10 points) Find the error in the following proof that $2 = 1$.

   Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$; subtract b^2 from both sides to get $a^2-b^2 = ab-b^2$; factor each side to get $(a+b)(a-b) = b(a-b)$; divide by $(a-b)$ to get $a+b = b$; let $a = b = 1$; the result is $2 = 1$

9. (20 points) Find the error in the following proof that all horses are the same color.

   Claim: In a set of n horses, all are of the same color.
   Proof: by induction on n.

   - **Basis:** For $n = 1$ the set contains just one horse and hence all horses are of the same color.
   - **Induction step:** For $n > 1$ assume that the claim is true for $n = k$ and prove that it is true for $n = k+1$. Take any set set H of $k+1$ horses. We show that all the horses in this set have the same color. For that we remove one horse, h_1 from H thus obtaining the set H_1 of k horses, which have the same color. Now, bring back in H_1 the horse h_1 and remove another horse from H_1 thus obtained, different from h_1, thus obtaining the set H_2. By the same argument all horses in H_2 have the same color. Hence, all horses in H must have the same color. what completes the proof.

10. (30 points) Show that every simple graph (for each node $n \in G$, no edge (n, n) allowed) with 2 or more nodes contains two nodes that have equal degrees.

11. (40 points) **Ramsey's theorem:** Let G be a graph. A *clique* in G is a subgraph of G in which every two nodes are connected by an edge. An *anti-clique*, also called an *independent set*, is a subgraph of G in which every two nodes are not connected by an edge. Show that every graph with n nodes contains either a clique or an anti-clique with at least $\frac{1}{2}log_2 n$ nodes (40 points).

12. **Theorem:** for each $t \geq 0$, let P_t be the amount of the loan outstanding after t month, Y be the monthly payment, I be the yearly interest rate of the loan and $M = 1 + I/12$ be the monthly multiplier. Then P_t is computed by the formula:

    $$P_t = PM^t - Y\frac{M^t - 1}{M - 1}$$

    (20 points) You are required to derive a formula for calculating the size of the monthly payment for a mortgage in term of the principal P, interest I, and the number of payments t. Assume that, after t payments have been made, the loan amount is reduced to 0. Use the formula to calculate the dollar amount of each monthly payment for a 30-year mortgage with 360 monthly payments on an initial loan amount of $\$100,000$ with 5% annual interest rate.

Due date is Wednesday 16 September 2009.