Mathematical Notions and Terminology [a]

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Set Theory

Set theory is used as mathematical foundation for many fields of mathematics and its applications, including Theory of Computation.

Hence, we start this class with a presentation of main concepts in set theory.

Sets

- A *set* is a collection of objects represented as a unit.
- The objects in a set are called *elements* or *members*.
- Sets may be described formally by:
  1. enumerating their elements
     Notation: $A = \{e_1, \ldots, , e_n\}$
  2. providing a property satisfied by all elements
     Notation: $A = \{x | P(x) = true\}$.

Example sets

- $S_1 = \{7, 21, 57\}$ is a set specified by the enumeration of its elements which are numbers
- $S_2 = \{$ student | student registered for 22C:xxx $\}$ is the set of students specified by the property “registered for 22C:xxx"

Notation

- The symbol $\in$ denotes set membership
- **Examples:** $7 \in S_1$ or $7 \in \{7, 21, 57\}$; $John\ Smith \in S_2$ or $John\ Smith \in \{$student | student registered for 22C:xxx $\}$
- The symbol $\notin$ denotes set non-membership
- **Example:** $5 \notin S_1$ or $5 \notin \{7, 21, 57\}$; $John\ Doe \notin S_2$ or $John\ Doe \notin \{$ student | student registered for 22C:xxx, Fall 2004 $\}$

Subsets

- For two sets A and B, we say that A is equal with B and write $A = B$, if every member of A is a member of B and every member of B is a member of A.
- For two sets, A and B, we say that A is a subset of B and write $A \subseteq B$, if every member of A is a member of B.
- We say that A is a *proper subset of* B and write $A \subset B$, if A is a subset of B and A is not equal to B, $A \neq B$.

Examples set equality

- $\{7, 21, 57\} = \{21, 7, 57\}$
- $\{7, 57\} \subset \{21, 7, 57\}$
- $\{7, 21, 57\} \subseteq \{21, 7, 57\}$
- $S_1 \neq S_2$, $S_1 \not\subset S_2$, $S_1 \not\subseteq S_2$

Rule: if A and B are sets, to prove their equality, i.e., to prove $A = B$ one needs to show two things:
(1) $\forall a \in A \Rightarrow a \in B$ and
(2) $\forall b \in B \Rightarrow b \in A$.

Facts

1. The order of elements while describing a set does not matter, nor does the repetition of its members.
2. **Example:** we get the same set by writing $\{57, 7, 7, 7, 21\}$ or $\{57, 7, 21\}$.
3. **Multiset:** a multiset is a collection of objects where number of occurrences of its members is taken into account.
4. **Example:** $\{7, 7, 7\}$ and $\{7\}$ are equal sets but they are different multisets.

Infinite sets

- An infinite set contains infinite many elements.
- We cannot write a list of all elements of an infinite set; so an infinite set is specified by the notation $S = \{e|p(e)\}$.
- **Examples:**
  - The set of natural numbers is written $\mathcal{N} = \{1, 2, 3, \ldots\}$ where $\ldots$ denotes the property “any element is equal to its predecessor plus 1".
  - The set of integer numbers is written $\mathcal{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ where $\ldots$ denotes the property “any element is equal to its predecessor plus 1".

How can we measure the size of a set?

Empty set

- A set with 0 members is called the *empty set* and is denoted by $\emptyset$.
- Axiomatically we can construct all sets from the set $\emptyset$ using set definition mechanisms.

Set construction

- By definition, there is a set that has no elements; this is the empty set denoted by $\emptyset$.
- A set can be constructed by enumerating its elements. If x_1, x_2, x_3 are given objects then $\{x_1, x_2, x_3\}$ is the set of these objects.
- A set can be constructed by giving a property satisfied by its elements. For example, the set of even numbers is $\{x | x\%2 = 0\}$.

Subset

If A is a set then $B = \{x|x \in A\}$ is a set called a subset of A and is denoted $B \subseteq A$.

Operations with sets

If A, B are sets then:

- $A \cup B = \{x | x \in A \vee x \in B\}$ is set union;
- $A \cap B = \{x | x \in A \wedge x \in B\}$ is set intersection;
- $A \setminus B = \{x | x \in A \wedge x \notin B\}$ is complement of B relative to A or set difference.

Special sets

- **Power set:**

  if A is a set then $\mathcal{P}(A) = \{B | B \subseteq A\}$ is the power set of A;

- **Example:**

  If $A = \{0, 1\}$s then $\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

- **Set of natural numbers, $\mathcal{N}$:**

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{\emptyset\} = \{0\} \\ 2 &= \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\ 3 &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\ \ldots \end{aligned}$$

Special properties

- **Inductive set:**

  Let A be a set and $succ(A) = A \cup \{A\}$. Then A is inductive if:

  1. $\emptyset \in A$ and
  2. $\forall a \in A \Rightarrow succ(a) \in A$.

  Example: the set $\mathcal{N}$ of natural numbers is inductive.

- **Transitive set:**

  A is transitive iff $\forall a \in A, x \in a \Rightarrow x \in A$.

  Example: the set $\mathcal{N}$ of natural numbers is transitive.

Ordered pairs

- For x, y set elements, (x, y) is an ordered pair if $\forall(u, v)$ we have:
  1. $(x, y) = (u, v) \Rightarrow x = u, y = v$ and
  2. $x = u, y = v \Rightarrow (x, y) = (u, v)$.

Set representation: the ordered pair (x, y) is represented as a set by $(x, y) = \{\{x\}, \{x, y\}\}$.

Quiz: show that $\{\{x\}, \{x, y\}\}$ is the set representation of the pair (x, y).

Relations on a set A

A relation on a set A is a set of ordered pairs of elements of A.

Examples: $<$ on natural numbers defined by:
$< = \{(x, y) | x, y \in \mathcal{N} \wedge x < y\}$

Notation: $(x, y) \in R$ is usually denoted by xRy;

For example: $(x, y) \in <$ is denoted $x < y$

Properties of relations

- R is transitive: $xRy \wedge yRz \Rightarrow xRz$;
- R is reflexive: xRx;
  Note: $x < x$ is not true on $\mathcal{N}$, i.e., $<$ is not reflexive.
- R is symmetric: $xRy \Rightarrow yRx$;
  Example: equality on $\mathcal{N}$: $x = y$ imply $y = x$.
- R is antisymmetric: $xRy \wedge yRx \Rightarrow x = y$;
  Example: $\leq$ on $\mathcal{N}$: $n \leq m$ and $m \leq n$ implies $n = m$.

Notation

If R is a relation on A then:

- $dom(R) = \{x \in A | \exists y \in A \wedge (x, y) \in R\}$ is called the domain of R;
- $ran(R) = \{y \in A | \exists x \in A \wedge (x, y) \in R\}$ is called the range of R.

Special relations

- *Total order:* R is a total order on A if
  1. R satisfies trichotomy, i.e., $\forall x, y \in A$ $x = y$ or xRy or yRx and
  2. R is transitive.

  Example: $<$ is total on $\mathcal{N}$
- *Partial order:* R is a partial order if it is transitive and irreflexive, i.e., it is never the case that xRx. **Example:** $<$ is partial order on $\mathcal{N}$

Another special relation

- *Equivalence:* R is an equivalence iff it is transitive, symmetric, and reflexive.

  Example: $\leq$ is an equivalence on $\mathcal{N}$.

A potential confusion

- Maclane definition of partial order:

  R is a partial order when it is reflexive, transitive, and antisymmetric.

- Enderton definition of partial order:

  R is a partial order when it is transitive and irreflexive

- How can we reconcile the difference?

- Irreflexive: R is irreflexive iff xRx for no x;
- Antisymmetric: R is antisymmetric iff xRy and yRx implies $x = y$;
- Asymmetric: R is asymmetric iff $xRy \Rightarrow not(yRx)$, i.e. R is both irreflexive and antisymmetric.

Note: for transitive relations asymmetry is equivalent to irreflexivity.

Observations

- When R is transitive and reflexive mathematicians usually call it a preorder;
- When R is transitive and irreflexive, mathematicians usually call it a strict partial order.

Venn diagrams

- Venn diagrams are visual pictures used to clarify concepts;
- A Venn diagram represents sets as regions enclosed by circular lines;
- There is much more about Venn diagrams! Use Google to find all about it!

Examples:

1. Figure 1 represents the set of English words starting with “t";
2. Figure 2 represents the set of English words ending with “z".

Example Venn diagram

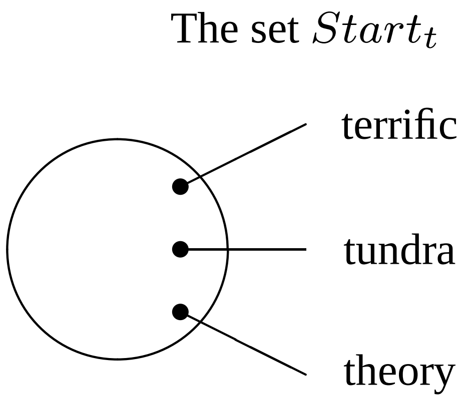

Figure 1: English words starting with “t"

Another Venn diagram

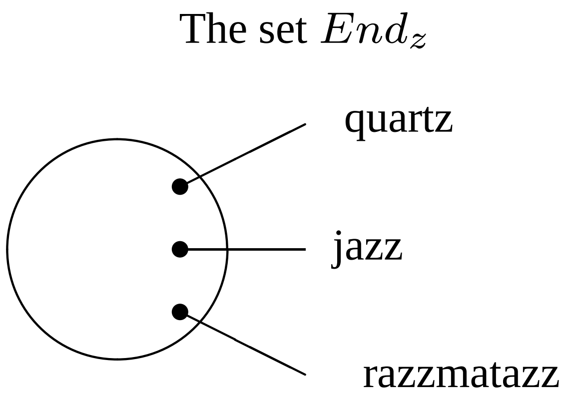

Figure 2: English words ending with “z"

Using Venn diagrams

- To represent two or more sets in the same Venn diagram we must draw them so they overlap indicating that the sets represented share some elements.

Examples:

1. Figure 3 represents the sets *Star-t*, *End-z*, *Start-j*
2. One can see that $topazz \in Start_t \cap End_z$ and $jazz \in Start_j \cap End_z$ and $Start_t \cap Start_j = \emptyset$

Using Venn diagrams

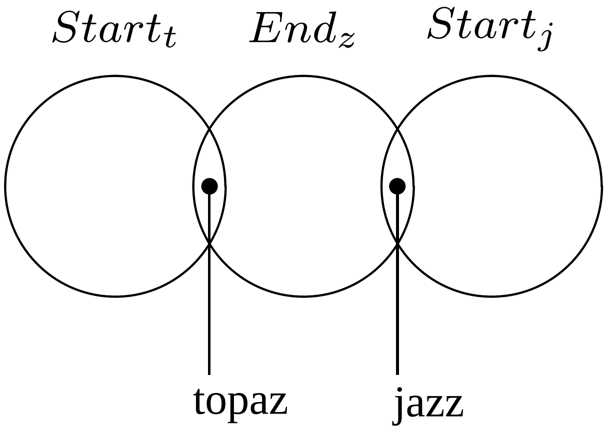

Figure 3: Venn diagram for common elements

Union and intersection

Figure 4 shows the Venn diagrams of sets obtained by operations set union and set intersection.

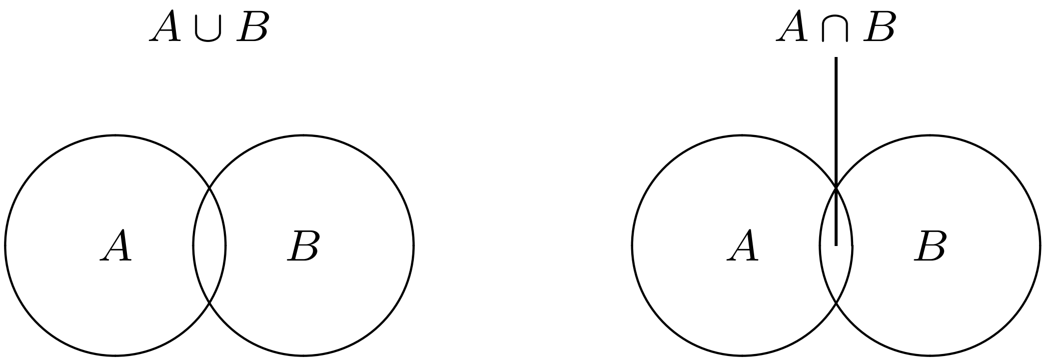

Figure 4: Venn diagrams for $A \cup B$ and $A \cap B$

Sequences

- A *sequence* of objects is a list of objects in some order.
- **Notation:** a sequence is usually denoted by writing the list within parenthesis.
- **Example:** the sequence 7, 21, 57 would be written $(7, 21, 57)$.

Facts

1. In a set the order of elements doesn't matter, but in a sequence it does. Hence $(7, 21, 57)$ is not the same with $(57, 21, 7)$.
2. Repetition is not permitted in a set but it is allowed in a sequence; hence we have:
    - the lists $(7, 7, 21, 57) \neq (7, 21, 57)$;
    - the sets $\{7, 21, 57\} = \{7, 7, 21, 57\}$.
3. As with sets, sequence may be finite or infinite.

Tuples

- Finite sequences often are called *tuples*;
- A sequence with k elements is a $k - tuple$;
- A $2 - tuple$ is called a *pair*.

Example:

1. $(7, 21, 57)$ is a $3 - tuple$

Facts

1. Sets and sequences may appear as elements of other sets and sequences;
2. **Example:** The *power set* of a set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A;
3. If $A = \{0, 1\}, \mathcal{P}(A) = \{\emptyset, 0, 1, \{0, 1\}\}$;
4. The set of all pairs of elements of A is $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Cartesian product

Cartesian product or *cross product* of the sets A and B, written $A \times B$, is the set of all pairs wherein the first element is a member of A and the second element is a member of B.

Example: if $A = \{1, 2\}$ and $B = \{x, y, z\}$ then $A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}$.

Cartesian product of k sets

- $A_1 \times A_2 \times \ldots \times A_k = \{A_1 \times A_2 \times \ldots \times A_{k-1}\} \times A_k$.
- **Example:** $A_1 \times A_2 \times A_3 = \{A_1 \times A_2\} \times A_3$.
- $A \times B \times A = \{(1,x,1),(1,x,2),(1,y,1),(1,y,2),(1,z,1),(1,z,2),$ $(2,x,1),(2,x,2),(2,y,1),(2,y,2),(2,z,1),(2,z,2)\}$.

Notation: $A \times A \times \ldots \times A$ taken k-times is denoted A^k.

Axiomatic set theory

- The (naive) set theory was created principally by the German mathematician Georg Cantor at the end of the 19th century.
- Set theory has come to play the role of a foundational theory in modern mathematics.

  It interprets propositions about mathematical objects (numbers, functions, etc.,) from all the traditional areas of mathematics (algebra, analysis, topology, etc.) in a single theory, and provides a standard set of axioms to prove or disprove them (see Bourbaky’s work).
- Axiomatic set theory is a rigorous reformulation of set theory in first-order logic, created to address paradoxes in naive set theory.

Observations

- The basic concepts of set theory are used throughout mathematics and computer science.
- Hence, it is advisable that computer science students to be exposed to the axiomatic set theory.

Basic concepts

Set and membership:

- A set is thought of as any collection of objects, called the members (or elements) of the set.
- The members of sets are any mathematical objects, and in particular can themselves be sets.

Example sets:

1. the set $\mathcal{N}$ of natural numbers 0, 1, 2, 3, 4, ... ,
2. the set $\mathcal{R}$ of real numbers,
3. the set $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{N}$ of functions from the natural numbers to the natural numbers,
4. the set $\{0, 2, N\}$ which has as members the numbers 0 and 2 and the set N.

Observations

- Initially, what is now known as "naive" or "intuitive" set theory was developed.
- However, assuming that one could perform any operations on sets without restriction led to paradoxes such as Russell’s paradox.

Russell's paradox

Let R be "the set of all sets that do not contain themselves as members".

Formally: A is an element of R if and only if A is not an element of A, i.e. $R = \{A | A \notin A\}$. Is R an element of itself?

Note: according to this definition if $R \in R$ then $R \notin R$; if $R \notin R$ then $R \in R$.

To address this problem set theory had to be re-constructed, using an axiomatic approach.

The history

- One particular set of axioms for set theory, put in their final form by Skolem, is called the Zermelo-Fraenkel set theory (ZF).
- Actually, this term usually excludes the axiom of choices (see below), which was once more controversial than it is today. When this axiom is included, the resulting system is called ZFC (Zermelo-Franked with Choice).
- An important feature of ZFC is that every object that it deals with is a set. In particular, every element of a set is itself a set.
- Other familiar mathematical objects, such as numbers, must be subsequently defined in terms of sets.

The axioms

1. **Axiom of empty set:** There is a set with no elements.
   We will use $\{\}$ or $\emptyset$ to denote this empty set.
2. **Axiom of extensionality:** Two sets are the same if and only if they have the same elements.
3. **Axiom of pairing:** If x, y are sets, then so is $\{x, y\}$, a set containing x and y as its only elements.
4. **Axiom of union:** Every set has a union. That is, for any set x there is a set y whose elements are precisely the elements of the elements of x.
5. **Axiom of infinity:** There exists a set x such that $\{\}$ is in x and whenever y is in x, so is the union $y \cup \{y\}$.
6. **Axiom of separation (or subset axiom):** Given any set X and any proposition $P(x), x \in X$, there is a subset of the original set X containing precisely those elements $x \in X$ for which $P(x)$ holds.

More axioms

7. **Axiom of replacement:** Given any set and any mapping, formally defined as a proposition $P(x, y)$ where $P(x, y)$ and $P(x, z)$ implies $y = z$, there is a set containing precisely the images of the original set's elements.

8. **Axiom of power set:** Every set has a power set. That is, for any set x there exists a set y, such that the elements of y are precisely the subsets of x.

9. **Axiom of regularity (or axiom of foundation):** Every non-empty set x contains some element y such that x and y are disjoint sets.
   More intuitive: Every nonempty set is disjoint of its elements!

10. **Axiom of choice: (Zermelo's version)** Given a set x of mutually disjoint nonempty sets, there is a set y (a choice set for x) containing exactly one element from each member of x.

Observations

- The axioms of choice and regularity are still controversial among a minority of mathematicians.
- Other axiom systems for set theory are Von Neumann-Bernays-Gödel set theory (NBG), the Kripke-Platek set theory (KP), etc;
- All of these are axiomatic set theories closely related to ZFC;
- Axiomatic set theories with quite different approaches are (for example) New Foundations and systems of positive set theory.

Note: use Google to find more about axiomatic set theory