Frequently Used Concepts in Computation Theory

Following the text Logic for Mathematicians *by A.G.Hamilton, Cambridge University Press 1988.*

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Propositional logic

Propositions: are natural language expressions constructed from the language vocabulary and few connectives (called here constructors) which will be introduced progressively and inductively as needed.

Inductive definition of propositions:

1. Every term of the vocabulary is a proposition and its interpretation is given in the language dictionary.
2. If $p_1, \ldots, p_n$ are propositions and c is an n-ary constructor (i.e., a proposition constructor that takes n arguments) then $c(p_1, \ldots, p_n)$ is well formed language expression (i.e., a proposition) whose interpretation depends upon the interpretations of $p_1, \ldots, p_n$ and the constructor c.

Assumption

The interpretations of propositions are values of true or false, called truth values, denoted further by T and F, respectively.

Note:

- T and F are two distinct language elements and their meaning is conventionally established and is clear for all language users.
- We assume here that the meaning of a proposition p can be computed by a function $f_c(p)$ which can be applied inductively on the components of p as we will show further.
- The computation performed by f_c can be represented by a table called a *truth table*.

Constructors and interpretations

Negation, $\neg$: if p is a proposition then $\neg p$ is a new proposition whose interpretation is obtained from the interpretation of p using the following truth table:

p	$\neg p$
T	F
F	T

Hence, the function $f_\neg(p)$ is defined by the equality:

$$f_\neg(p) = \begin{cases} F & \text{if } p = T, \\ T & \text{if } p = F. \end{cases}$$

Conjunction, $\wedge$

If p_1, p_2 are propositions then $p_1 \wedge p_2$ is a new proposition whose interpretation is obtained from the interpretations of p_1 and p_2 using the following truth table:

p_1	p_2	$p_1 \wedge p_2$
T	T	T
T	F	F
F	T	F
F	F	F

f_λ

Hence, the function $f_\wedge(p_1, p_2)$ is defined by the equality:

$$f_\wedge(p_1, p_2) = \begin{cases} T & \text{if } p_1 = T \text{ and } p_2 = T, \\ F & \text{if } p_1 = T \text{ and } p_2 = F, \\ F & \text{if } p_1 = F \text{ and } p_2 = T, \\ F & \text{if } p_1 = F \text{ and } p_2 = F. \end{cases}$$

Disjunction, $\vee$

If p_1, p_2 are propositions then $p_1 \vee p_2$ is a new proposition whose interpretation is obtained from the interpretations of p_1 and p_2 using the following truth table:

p_1	p_2	$p_1 \vee p_2$
T	T	T
T	F	T
F	T	T
F	F	F

$f_\vee$

Hence, the function $f_\vee(p_1, p_2)$ is defined by the equality:

$$f_\vee(p_1,p_2) = \begin{cases} T & \text{if } p_1 = T \text{ and } p_2 = T, \\ T & \text{if } p_1 = T \text{ and } p_2 = F, \\ T & \text{if } p_1 = F \text{ and } p_2 = T, \\ F & \text{if } p_1 = F \text{ and } p_2 = F. \end{cases}$$

Implication, $\rightarrow$

If p_1, p_2 are propositions then $p_1 \rightarrow p_2$ is a new proposition whose interpretation is obtained from the interpretations of p_1 and p_2 using the following truth table:

p_1	p_2	$p_1 \rightarrow p_2$
T	T	T
T	F	F
F	T	T
F	F	T

$f_{\rightarrow}$

Hence, the function $f_{\rightarrow}(p_1, p_2)$ is defined by the equality:

$$f_{\rightarrow}(p_1, p_2) = \begin{cases} T & \text{if } p_1 = T \text{ and } p_2 = T, \\ F & \text{if } p_1 = T \text{ and } p_2 = F, \\ T & \text{if } p_1 = F \text{ and } p_2 = T, \\ T & \text{if } p_1 = F \text{ and } p_2 = F. \end{cases}$$

Equivalence, $\leftrightarrow$

If p_1, p_2 are propositions then $p_1 \leftrightarrow p_2$ is a new proposition whose interpretation is obtained from the interpretations of p_1 and p_2 using the following truth table:

p_1	p_2	$p_1 \leftrightarrow p_2$
T	T	T
T	F	F
F	T	F
F	F	T

$f_{\leftrightarrow}$

Hence, the function $f_{\leftrightarrow}(p_1, p_2)$ is defined by the equality:

$$f_{\leftrightarrow}(p_1, p_2) = \begin{cases} T & \text{if } p_1 = T \text{ and } p_2 = T, \\ F & \text{if } p_1 = T \text{ and } p_2 = F, \\ F & \text{if } p_1 = F \text{ and } p_2 = T, \\ T & \text{if } p_1 = F \text{ and } p_2 = F. \end{cases}$$

Statement form

A statement form S is an expression involving statement variables (i.e., variable denoting statements) and constructors which can be generated using the rules:

(i) Any statement variable is a statement form.

(ii) If S_1, S_2 are statement forms then $(\neg S_1)$, $(S_1 \wedge S_2)$, $(S_1, \vee S_2)$, $(S_1 \rightarrow S_2)$, and $(S_1 \leftrightarrow S_2)$ are statement forms.

Statement form interpretation

- Using truth tables for the constructors employed to generate a statement form we can construct a truth table of every statement form S.
- This truth table will indicate, for any given assignment of the truth values to the statement variables appearing in S, the truth value S takes.
- Hence, each statement form gives rise to a truth function of a number of arguments equal to the number of different statement variables appearing in that statement form.

Example

Consider statement form $((\neg p) \vee q)$

Its truth table is:

p	q	$(\neg p)$	$((\neg p) \vee q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Tautology and contradiction

Tautology: is a statement form S which takes the truth value T under each possible assignment of truth values to the statement variables which occur in S

Contradiction: is a statement form S which takes the truth value F under each possible assignment of truth values to the statement variables which occur in S

Examples

(a) $(p \vee (\neg p))$ is a tautology.

(b) $(p \wedge (\neg p))$ is a contradiction.

(c) $(((\neg p) \longrightarrow q) \longrightarrow (((\neg p) \longrightarrow (\neg q)) \longrightarrow p))$ is a tautology.

Proof

To prove these statements one needs to construct the truth tables of the statement forms involved.

We construct here only the truth table of statement form (b).

p	$(\neg p)$	$(p \wedge (\neg p))$
T	F	F
F	T	F

Logical equivalence

- If S_1 and S_2 are statement forms, S_1 logically implies S_2 if $(S_1 \longrightarrow S_2)$ is a tautology;
- S_1 is logically equivalent to S_2 if $(S_1 \leftrightarrow S_2)$ is a tautology.

Examples

(a) $(p \wedge q)$ logically implies p

(b) $(\neg(p \wedge q))$ is logically equivalent to $((\neg p) \vee (\neg q))$

(c) $(\neg(p \vee q))$ is logically equivalent to $((\neg p) \wedge (\neg q))$

Proof

We show here (a) only by constructing the truth table of $((p \wedge q) \rightarrow p)$:

p	q	$(p \wedge q)$	$((p \wedge q) \rightarrow p)$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Statement form manipulation

Proposition 1: If S_1 and $(S_1 \longrightarrow S_2)$ are tautologies then S_2 is a tautology.

Proof: by contradiction.

- Suppose that S_1 and $(S_1 \rightarrow S_2)$ are tautologies and S_2 is not.
- Then there is an assignment of truth values to the statement variables that appear in S_1 or S_2 which gives S_2 value F.
- This same assignment must give S_1 the value T because S_1 is a tautology.
- Consequently $(S_1 \rightarrow S_2)$ has the value F, which is a contradiction.

Hence, S_2 must be a tautology.

Substitution

Let S be a statement form in which statement variables $p_1, \ldots, p_n$ occur, and let $S_1, \ldots, S_n$ be any statement forms. If S is a tautology then the statement S' obtained from S by replacing each occurrence of p_i by S_i, $1 \leq i \leq n$, throughout S, is also a tautology.

Proof: by construction

- Assign any truth values to statement variables which occur in $S_1, \ldots, S_n$.
- The truth value S' takes is the same as S would have taken if values which $S_1, \ldots, S_n$ take had been assigned to $p_1, \ldots, p_n$.
- Since S is a tautology this value is T.

That is, S' is a tautology.

Duality

Let S be a statement form containing only constructors $\neg$, $\wedge$ and $\vee$. Suppose that S^* is obtained from S by interchanging $\wedge$ and $\vee$ and replacing every statement variable by its negation, throughout S. Then S^* is logically equivalent to $\neg S$.

Proof

By induction on the number n of constructors used in S.

Base: If $n = 0$, S consists of a single variable, say p and S^* is $\neg p$. Then obviously $((\neg p) \leftrightarrow (\neg p))$ is a tautology. Thus S^* is logically equivalent to $(\neg S)$.

Step: Suppose that proposition is true for an S that contain n constructors and show that then it is true for S when it contain $n + 1$ constructors.

Note: due to the inductive construction, a statement form S that contain $n + 1$ constructors has one of the forms: $(\neg S_1)$, $(S_1 \vee S_2)$, $(S_1 \wedge S_2)$. The logical equivalence of S^* and $(\neg S)$ is shown for each of these cases.

Corollary

If $p_1, \ldots, p_n$ are statement variables then $((\neg p_1) \vee \ldots \vee (\neg p_n))$ is logically equivalent to $(\neg(p_1 \wedge \ldots \wedge p_n))$.

Notation: $(\bigvee_{i=1}^{i=n}(\neg p_i))$ is logically equivalent to $(\neg(\bigwedge_{i=1}^{i=n} p_i))$.

De Morgan's Laws:

Let $S_1, \ldots, S_n$ be statement forms. Then:

(i) $(\bigvee_{i=1}^{n}(\neg S_i))$ is logically equivalent to $(\neg(\bigwedge_{i=1}^{n} S_i))$

(ii) $(\bigwedge_{i=1}^{n}(\neg S_i))$ is logically equivalent to $(\neg(\bigvee_{i=1}^{n} S_i))$.

Disjunctive normal form

Every statement form which is not a *contradiction* is logically equivalent to a statement form of the form $(\bigvee_{i=1}^{m}(\bigwedge_{j=1}^{n} Q_{ij}))$ where each Q_{ij} is either a statement variable or the negation of a statement variable.

Conjunctive normal form

Every statement form which is not a *tautology* is logically equivalent to a statement form of the form $(\bigwedge_{i=1}^{m}(\bigvee_{j=1}^{n} Q_{ij}))$ where each Q_{ij} is either a statement variable or the negation of a statement variable.

Adequate sets of constructors

Definition: an adequate set of constructors is a set such that every truth function can be represented by a statement form containing only the constructors from that set.

Proposition

The pairs $\{\neg, \wedge\}$, $\{\neg, \vee\}$ and $\{\neg, \rightarrow\}$ are adequate set of constructors.

Proof: by construction.

1. For any statement forms S_1 and S_2 $(S_1 \vee S_2)$ is logically equivalent to $(\neg((\neg S_1) \wedge (\neg S_2)))$. So, and statement form that contain just $\{\neg, \wedge, \vee\}$ can be transformed into a logically equivalent statement containing only $\neg$ and $\wedge$.
2. Similarly we can use the logical equivalence of $(S_1 \wedge S_2)$ with $(\neg((\neg S_1) \vee (\neg S_2)))$ to see that $\{\neg, \vee\}$ is adequate.
3. Since $(S_1 \wedge S_2)$ is logically equivalent to $(\neg(S_1 \rightarrow (\neg S_2)))$ and $(S_1 \vee S2)$ is logically equivalent to $((\neg S_1 \rightarrow S_2))$ we can express any statement form only using constructors $\neg$ and $\rightarrow$.

Formal system

A formal system is specified by:

1. An alphabet of symbols
2. A set of well-formed formulas (wf) (words and sentences in our formal language)
3. A set of well-formed formulas called axioms
4. A finite set of “deduction rules" which enable one to deduce a wf S as a “direct consequence" of a set of wf-s $S_1, \ldots, S_n$.

Example formal system

The formal system L of statement calculus is defined as follows:

1. The alphabet is: $\neg$,$\rightarrow$,(,),$p_1, p_2, p_3, \ldots$
2. Set of wfs defined by:
   (a) p_i is a wf for each $i \geq 1$
   (b) If S_1 and S_2 are wfs then $(\neg S_1)$ and $(S_1 \rightarrow S_2)$ are wfs
   (c) The set of all wfs is generated by (i) and (ii).
3. The axioms are specified by the following axiom schemes:
   (L1) $(S_1 \rightarrow (S_2 \rightarrow S_1))$
   (L2) $((S_1 \rightarrow (S_2 \rightarrow S_3)) \rightarrow ((S_1 \rightarrow S_2) \rightarrow (S_1 \rightarrow S_3)))$
   (L3) $(((\neg S_1) \rightarrow (\neg S_2)) \rightarrow (S_2 \rightarrow S_1))$
4. Deduction rules: from S_1 and $(S_1 \rightarrow S_2)$, S_2 is a direct consequence, where S_1 and S_2 are wfs.

   Note: This is called Modus Ponens, (MP).

Proof

A proof in L is a sequence of wfs $S_1, \ldots, S_n$ such that for each i, $(1 \leq i \leq n)$, either S_i is an axiom of L or S_i follows from two previous members of the sequence, say S_j and S_k, $(j < i, k < i)$, as a direct consequence using MP.

Note: such a proof is referred to as the “proof of S_n in L" and S_n is called a theorem of L.

Observations

Let $S_1, \ldots, S_n$ be a proof of S_n in L.

1. If S_n is a direct consequence of S_j and S_k then S_j and S_k must be of the form P and $(P \rightarrow Q)$ or vice versa, for P,Q wfs;
2. If $S_1, \ldots, S_n$ is a proof and $k < n$ then $S_1, \ldots, S_k$ is a proof;
3. Axioms of L are theorems and their proofs are one-member sequences.

Deduction form

Let Γ be a set of wfs of L (which may or may not be theorems). A sequence $S_1, \ldots, S_n$ of wfs of L is a *deduction form* Γ if for each i, $1 \leq i \leq n$, one of the following hold:

(a) S_i is an axiom of L

(b) S_i is a member of Γ

(c) S_i follows from previous members of the sequence using MP

Observations

- A deduction from Γ is just a proof in which members of Γ are considered as temporary axioms.
- The last member, S_n, of a deduction from Γ is called a *consequence* of Γ in L.
- If S is a consequence of Γ in L we also say that Γ *yields* S and write it $\Gamma \vdash_L S$.
- A theorem S in L is a deduction from $\emptyset$, i.e. $\emptyset \vdash_L S$
- Note that $\vdash$ is not a symbol of L so any expression using it is not in L. Hence, a proof is not part of L, it is a statement about L.

Deduction theorem

If $\Gamma \cup \{S_1\} \vdash_L S_2$ then $\Gamma \vdash_L (S_1 \rightarrow S_2)$ where S_1, S_2 are wfs of L and Γ is a set of wfs of L (possible empty).

Proof: by induction on the number of wfs in the deduction of S_2.

Base: The sequence has one member, hence it must be S_2 itself. Hence, (1) S_2 is an axiom or (2) $S_2 \in \Gamma \cup \{S_1\}$.

(a) If S_2 is an axiom then the following is a deduction $\Gamma \vdash_L (S_1 \rightarrow S_2)$:
S_2 (axiom of L), $(S_2 \rightarrow (S_1 \rightarrow S_2))$ (L1), $(S_1 \rightarrow S_2)$ (by MP).

(b) If $S_2 \in \Gamma \cup \{S_1\}$ then $S_2 \in \Gamma$ or S_2 is S_1.
Case 1: $S_2 \in \Gamma$ then the following is a deduction of $S_1 \rightarrow S_2$:
S_2 (member of Γ), $(S_2 \rightarrow (S_1 \rightarrow S_2))$ (L1), $(S_1 \rightarrow S_2)$ (by MP).

Proof, continuation

(b) Continuation:

Case 2: S_2 is S_1. The following is a deduction of $S_1 \to S_1$:

Consider the axiom (L2): $((S_1 \to (S_2 \to S_3)) \to ((S_1 \to S_2) \to (S_1 \to S_3)))$ and replace S_1 by S_1, S_2 by $S_1 \to S_1$, and S_3 by S_1. Thus we obtain:

(1) $(S_1 \to ((S_1 \to S_1) \to S_1)) \to ((S_1 \to (S_1 \to S_1)) \to (S_1 \to S_1))$ (L2)

(2) $(S_1 \to ((S_1 \to S_1) \to S_1))$ (L1)

(3) $((S_1 \to (S_1 \to S_1)) \to (S_1 \to S_1))$ (MP)

(4) $(S_1 \to (S_1 \to S_1)$ (S_1 is S_1, S_2 is S_1) (L1)

(5) $(S_1 \to S_1)$ (MP)

Step: Similar adding the case S_2 obtained from two previous wfs in the deduction by MP

Converse

If $\Gamma \vdash_L (S_1 \rightarrow S_2)$ then $\Gamma \cup \{S_1\} \vdash_L S_2$.

Corollary: *hypothetical syllogism:*

$$\{(S_1 \rightarrow S_2), (S_2 \rightarrow S_3)\} \vdash_L (S_1 \rightarrow S_3)$$

Valuation of L

A valuation of L is a function
$v : \{wfs\ of\ L\} \longrightarrow \{T, F\}$ such that:

(i) $v(S) \neq v(\neg S)$ for any wf S

(ii) $v(S_1 \rightarrow S_2) = F$ iff $v(S_1) = T$ and $v(S_2) = F$

Tautology: a wf S of L is a tautology is $v(S) = T$ for every valuation v.

Soundness theorem

Every theorem of L is a tautology

Proof: by induction on the number of wfs of L in the sequence that constitutes a proof of the theorem

Extension of L: is a formal system obtained by altering or enlarging the set of axioms that preserves theorems of L

Consistent extension: an extension of L is consistent if for no wf S of L are both S and $(\neg S)$ theorems of the extension.

Complete extension: an extension of L is complete if for each wfs S either S or $(\neg S)$ is a theorem in the extension.

Adequacy and decidability

Adequacy theorem: If S is a wf of L and S is a tautology then $\vdash_L S$.

Proof: by contradiction. Suppose S is a tautology and not a theorem. The extension L^* of L by including $(\neg S)$ as axiom is consistent and there is a valuation v such that $v(S) = T$ and $v(\neg S) = T$ which is a contradiction.

Decidability

L is decidable, i.e., there is an effective method for deciding, given any wfs S of L, whether S is a theorem of L.

Proof: Consider S as a statement form and construct it truth table. It is a theorem iff it is a tautology.

Quantifiers

Universal quantifier: the phrase “for all x" is called a universal quantifier and is denoted by $(\forall x)$.

Example use: $(\forall x)(A(x) \to M(x))$ meaning for ever x in the Universe of Discourse (UoD) $A(x) \to M(x)$; x is called a bound variable.

Existential quantifier: the phrase “there exists at least one x in UoD such that" is called an existential quantifier and is denoted by $(\exists x)$.

Example use: $(\exists x)(P(x) \wedge Q(x))$ means that exists an $x \in UoD$ that has both properties P and Q; x is called a bound variable.

First order language

A first order language is a language $\mathcal{L}$ whose alphabet contain the symbols:

- Variables $x_1, x_2, \ldots$
- Individual constants (possible none) $a_1, a_2, \ldots$
- Predicate letters (possible non) $\pi_1^{n_1}, \pi_2^{n_2}, \ldots$ of arities $n_1, n_2, \ldots$
- Function letters (possible non) $f_1^{n_1}, f_2^{n_2}, \ldots$ of arities $n_1, n_2, \ldots$
- Punctuation symbols (,), ,
- Connectives (constructors) $\neg$ and $\rightarrow$
- The quantifier $\forall$.

Terms

A term in $\mathcal{L}$ is defined as follows:

(i) Variables and individual constants are terms in $\mathcal{L}$

(ii) If f_i^n is a function letter in $\mathcal{L}$ and $t_1, \ldots, t_n$ are terms then $f_i^n(t_1, \ldots, t_n)$ is a term in $\mathcal{L}$.

(iii) The set of all terms of $\mathcal{L}$ is generated by (i) and (ii).

Example terms

Assume that alphabet of $\mathcal{L}$ contains:

Individual constants: a_1 (stands for 0)

Predicate symbols: π_1^2 (stands for =)

Functional symbols: f_1^1 (successor), f_1^2 (addition), f_2^2 (multiplication)

Then $f_1^2(x_1, x_2)$ and $f_2^2(x_1, x_2)$ are terms standing for

$x_1 + x_2$ and $x_1 x_2$, respectively.

Atomic formula

An atomic formula in $\mathcal{L}$ is defined by:

if π_j^k is a predicate letter in $\mathcal{L}$ and $t_1, \ldots, t_k$ are terms in $\mathcal{L}$ then $\pi_j^k(t_1, \ldots, t_k)$ is an atomic formula of $\mathcal{L}$.

Using previous example, $\pi_1^2(f_1^2(x_1, x_2), f_2^2(x_1, x_2))$ is an atomic formula interpreted as $x_1 + x_2 = x_1 x_2$.

Well-formed formulas of $\mathcal{L}$

(i) Every atomic formula of $\mathcal{L}$ is a wf of $\mathcal{L}$

(ii) If S_1, S_2 are wfs of $\mathcal{L}$ then $(\neg S_1)$, $(S_1 \rightarrow S_2)$ and $(\forall x_i)S_1$ where x_i is a variable, are wfs of $\mathcal{L}$.

(iii) The set of all wfs of $\mathcal{L}$ is generated by (i) and (ii).

Notation: formulas are usually denoted by Greek letters $\alpha, \beta, \ldots$.

Abbreviations

- $(\exists x_i)\alpha$ is an abbreviation for $(\neg((\forall x_i)(\neg\alpha)))$
- $(\alpha_1 \wedge \alpha_2)$ is an abbreviation for $(\neg(\alpha_1 \rightarrow (\neg\alpha_2)))$
- $(\alpha_1 \vee \alpha_2)$ is an abbreviation for $((\neg\alpha_1) \rightarrow \alpha_2)$
- Parentheses may be omitted when the scope of the symbols is clear. For example, $(\neg\alpha_1 \rightarrow \alpha_2)$ is an abbreviation for $((\neg\alpha_1) \rightarrow \alpha_2)$.
- Let α be a wf of $\mathcal{L}$. A term t if free for x_i in α if x_i does not occur free in α within the scope of a $(\forall x_j)$ where x_j is any variable in t.

Interpretation and valuation

An interpretation I of $\mathcal{L}$ is a non-empty set D_I (domain of I) together with a collection of distinguished elements $\bar{a}_1, \bar{a}_2, \ldots$, of D_I, a collection of distinguished relations $\bar{\pi}_1, \bar{\pi}_2, \ldots$, on D_I, a collection of distinguished functions $\bar{f}_1, \bar{f}_2, \ldots$, on D_i.

A valuation of $\mathcal{L}$ in I is a function v from the set of terms of $\mathcal{L}$ to the set D_I with the properties:

(i) $v(a_i) = \bar{a}_i$ for each individual constant a_i of $\mathcal{L}$

(ii) $v(f_i^n(t_1, \ldots, t_n)) = \bar{f}_i^n(v(t_1), \ldots, v(t_n))$ where f_i^n is any function letter in $\mathcal{L}$ and $t_1, \ldots, t_n$ are terms in $\mathcal{L}$.

i-equivalent valuations

Two valuations v and v' are i-equivalent if $v(x_j) = v'(x_j)$ for every $j \neq i$.

Note:

- Valuations which are i-equivalent have the same values on each of the variables, except possible on x_i but in general the values will differ on any term t which contains x_i.
- A given valuation v will assign a truth value to a wf α of $\mathcal{L}$ by replacing each term t of S by $v(t)$ and replacing each function and predicate letters by their interpretations in I.

  Note: what is obtained is a statement about elements of D_I which may be true or false. If it is true then v satisfies α.

Satisfaction and truth

Let α be a wf of $\mathcal{L}$ and I be an interpretation of $\mathcal{L}$. A valuation v in I is said to satisfy α if it can be shown inductively to do so under the following conditions:

(i) v satisfies the atomic formula $\pi_i^n(t_1, \ldots, t_n)$ if $\bar{\pi}_i^n(v(t_1), \ldots, v(t_n))$ is true in D_i.

(ii) v satisfies $(\neg\alpha)$ if v does not satisfy α

(iii) v satisfy $(\alpha_1 \rightarrow \alpha_2)$ if either v satisfies $(\neg\alpha_1)$ or v satisfies α_2.

(iv) v satisfies $(\forall x_i)\alpha$ if every valuation v' which is i-equivalent to v satisfies α.

Relationship between L and $\mathcal{L}$

- Let S_0 be a wf of L. If we replace each statement letter occurring in S_0 by a wf of $\mathcal{L}$ (replacing the same letter by the same wf throughout) we obtain a wf α of $\mathcal{L}$.
- α obtained above is called a *substitution instance* of S_0 in $\mathcal{L}$.
- Similarly, starting with a wf α_0 of $\mathcal{L}$ the wf obtained will have the same structure as some (usually more than one) wf S of L.

Example: $((\forall x_1)\pi_1^1(x_1) \rightarrow (\forall x_1)\pi_1^1(x_1))$ is a substitution instance of $(p_1 \rightarrow p_1)$.

Since $(p_1 \rightarrow p_1)$ is a tautology we can extend tautology notion to $\mathcal{L}$.

Tautology in $\mathcal{L}$

- A wf α of $\mathcal{L}$ is a tautology if it is a substitution instance in $\mathcal{L}$ of a tautology in L.
- A wf α of $\mathcal{L}$ which is a tautology is true in any interpretation of $\mathcal{L}$.
- A wf α of $\mathcal{L}$ is logically valid if it is true in every interpretation of $\mathcal{L}$; α is a contradiction if it is false in every interpretation of $\mathcal{L}$.
- A wf α of $\mathcal{L}$ is closed if no free variable occurs in α.

Model

Let Γ be a set of wfs of $\mathcal{L}$. An interpretation of $\mathcal{L}$ in which each element of Γ is true is called a model of Γ.

Note: if Δ is a first order system, a model of Δ is an interpretation of $\mathcal{L}$ in which every theorem of Δ is true.