22C : 231 **Design and Analysis of Algorithms**
Midterm Exam

The duration of this exam is one hour and fifteen minutes. This is a closed-book exam.

1. (10 points)

   (a) We are given a flow network G with vertex set V and a nonnegative integer capacity $c(u, v)$ for any edge $(u, v) \in V \times V$. We are also specified a source $s \in V$ and a sink $t \in V$. State the definition of a flow in the network G. Define the value of the flow.

   (b) Consider the figure on the next page. What is the value of the flow that is depicted?

   (c) Argue that flow conservation holds at v_3.

   (d) Argue that the flow depicted is a maximum flow.

   (e) In the figure, what is the capacity of the minimum-capacity s-t cut?

2. (10 points) We are given a sequence $S = \langle s_1, s_2, \ldots, s_n \rangle$ of n distinct positive integers. The sequence is not necessarily sorted. The problem is to develop an algorithm for finding the longest increasing subsequence of S. For example, if $S = \langle 100, 10, 60, 70, 20, 30, 40, 80 \rangle$, then $\langle 10, 20, 30, 40, 80 \rangle$ is the longest increasing subsequence. Let MAX be the largest integer in S. An algorithm whose running time is bounded by a polynomial in n and MAX is acceptable, though this will get slightly lesser credit than an algorithm whose running time is bounded by a polynomial in just n. One way to obtain such an algorithm is suggested below – you may ignore this suggestion and develop an efficient algorithm using different ideas. It is also enough if your algorithm outputs just the length of the longest increasing subsequence, and not the subsequence itself.

   For $1 \leq i \leq n$, let S^i denote the sequence $\langle s_1, \ldots, s_i \rangle$. For any integer t, let $\mu(i, t)$ denote the length of the longest subsequence among all increasing subsequences of S_i whose elements are strictly less than t. For instance, $\mu(5, 40) = 2$ whereas $\mu(5, 80) = 3$ in the above example. Note that $\mu(n, \text{MAX}+1)$ is the length of the longest increasing subsequence of S. For any $2 \leq i \leq n$, we have the following recurrence relation:

$$\begin{aligned}
\mu(i, t) &= \mu(i-1, t) \text{ if } s_i \geq t \\
\mu(i, t) &= \max\{\mu(i-1, t), \mu(i-1, s_i) + 1\} \text{ if } s_i < t.
\end{aligned}$$

   If you develop your algorithm based on this recurrence relation, give an informal justification for the recurrence relation.

3. (5 points) Consider a connected, undirected graph $G = (V, E)$. Define a cut in this graph as any subset of edges whose removal disconnects the graph. Let E' be a cut

in the graph with the smallest number of edges. We choose a random edge from E, where each edge from E is equally likely to be chosen. Argue that the probability that the chosen edge will belong to E' is at most $2/|V|$.