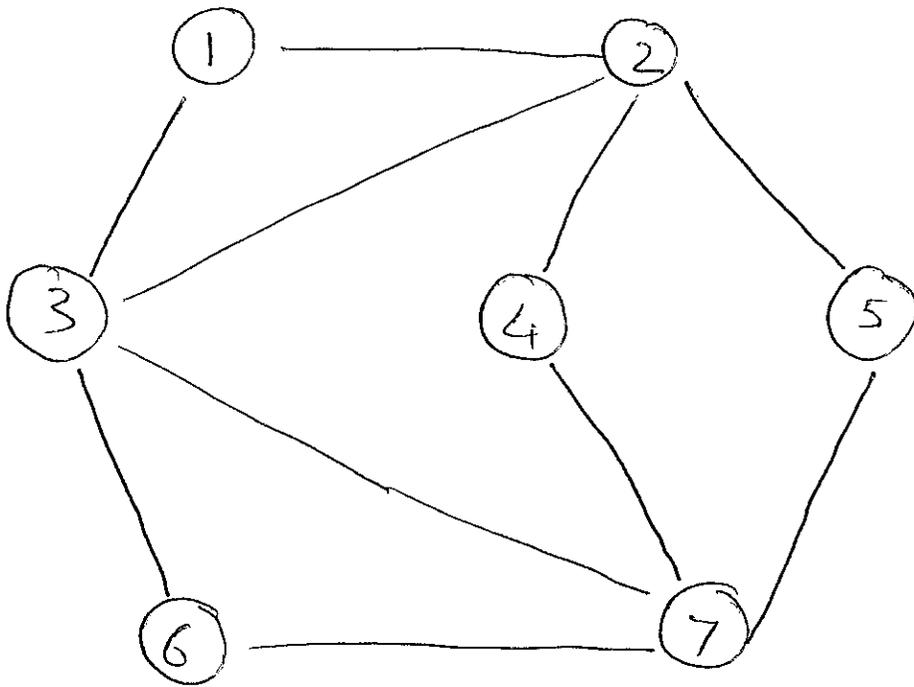


Polynomial-Time Reductions

The independent set problem:

Given an undirected graph $G = (V, E)$, a set $S \subseteq V$ of vertices is independent if no two vertices in S are joined by an edge in E .

The independent set problem is to determine, given a graph G and integer k , if G has an independent set of size at least k . Note that this is a decision version of an optimization problem that asks for the largest independent set.



$\{3, 4, 5\}$ is an independent set.

$\{1, 6, 4, 5\}$ is a bigger one.

~~The independent set problem is one about which we don't know~~

We don't know if the independent set problem can be solved in polynomial time. That is, we don't

know if there is a poly-time algorithm that solves the problem.

The Vertex Cover problem: ~~Given~~ In a graph $G=(V,E)$, a vertex cover is a subset $S \subseteq V$ such that each edge in E has at least one of its endpoints in S .

In the example graph, $\{1,2,6,7\}$ is a vertex cover. So is $\{2,3,7\}$.

The vertex cover problem asks, given a graph G and integer k' , if there is a vertex cover of size at most k' . (A decision version of a minimization problem.)

Well, we don't know if there is a polynomial algorithm for vertex cover either.

But we do know that if there is a polynomial algo for vertex cover, there is one for independent set as well. And vice versa.

This is based on the simple

Observation: $S \subseteq V$ is an independent set in $G = (V, E)$ if and only if $V - S$ is a vertex cover.

So $G = (V, E)$ has an independent set of size at least k iff

G has a vertex cover of size at most $|V| - k$.

Solving Independent set given a black box for vertex cover:

Suppose we want to know if G has an independent set of size $\geq k$. We ask the black box if G has a vertex cover of size at most $|V| - k$. If blackbox says yes, we return yes. If it says no, we return no.

Similarly, we can solve vertex cover given a black box for independent set.

Note that these algorithms are polynomial-time, but they ~~only~~ contain calls to the black boxes.

Notice that if there is a polynomial-time algo for vertex cover, then replacing the black box for vertex cover with this algo yields a polynomial time algo for independent set.

Def. Problem Y is polynomial-time reducible to problem X , denoted $Y \leq_p X$, if there is an algo that solves Y using a polynomial number of Computational steps, plus a polynomial no. of calls to a black box for solving X .

So Independent-set \leq_p Vertex-Cover

and Vertex-Cover \leq_p Independent-Set.

A Consequence of the defn:

① Suppose $Y \leq_p X$. If X can be solved in polynomial time, then Y can be solved in polynomial time as well.

② Suppose $Y \leq_p X$. If Y can't be solved in poly-time, then neither can X .

② is the ~~sp~~ direction in which we will exploit poly-time reducibility: we'll infer the hardness of a problem by reducing to it a known problem that's believed to be hard.

Let's do a more interesting example of a poly-time reduction...