

# The Gale-Shapley Algorithm

During the course of the algorithm, certain pairs  $(m, u) \in M \times W$  become "engaged". The set of engaged pairs forms a matching. We will say ~~say~~ that an individual is engaged to mean that he/she is part of a pair that is engaged. An individual is said to be free if he/she is not engaged.

Initially, all  $m \in M$  and  $w \in W$  are free.

While there is a man  $m$  who is free  
and hasn't proposed to every woman

Choose such a man  $m$

Let  $w \leftarrow$  highest-ranked woman in  
 $m$ 's list to whom  $m$  has  
not yet proposed.

$m$  makes a proposal to  $w$ :

If  $w$  is free then  $(m, w)$  become  
engaged.

Else Let  $m'$  be ~~woman~~ to whom  
 $w$  is currently engaged.

If  $w$  prefers  $m'$  to  $m$  then  
 $m$  remains free.

Else  $w$  prefers  $m$  to  $m'$ .

$(m, w)$  become engaged.

$(m', w)$  become disengaged,  
and so  $m'$  becomes free.

Endif

Endif

Endwhile

Return the set of engaged pairs.

## Analysis of Algorithm.

Observation 1: The G-S Algorithm terminates after at most  $n^2$  iterations of the while loop.

Proof: Within each while loop -

Some man  $m$  makes a proposal to some woman  $w$ . Observe that  $m$  has never made a proposal to  $w$  before, why?

This means that a given man  $m$  proposes to a given woman  $w$  at most once. There are exactly  $n^2$  man-woman pairs, so at most  $n^2$  proposals are made, so there are at most  $n^2$  iterations of while-loop.

The above observation hints that ~~a~~  
a bound on the number of steps  
that an implementation of the  
algorithm would take. Note that  
what is needed to implement the  
steps within ~~a~~ the while loop is  
basically a book-keeping mechanism.  
More on this later.

We now show ~~that~~ the less  
obvious facts that the G-S algorithm  
returns (a) a perfect matching, and  
(b) a stable matching.

Observation 2: Fix some woman  $w$ .  $w$   
remains engaged from the point at which  
she receives her first proposal. The  
sequence of partners to which she is  
engaged gets better and better (in her ordering).

Observation 3 Let  $m$  be any man. Suppose  $m$  is free just before the execution of the While statement. Then there is a woman to whom he has not yet proposed.

Proof: Suppose the conclusion is false and  $m$  has proposed to all women. Then by Observation 2, all women are currently engaged. Since the set of engaged pairs form a matching, this means ~~all~~ that  $n$  men, thus all men, are currently engaged. So  $m$  is also <sup>currently</sup> engaged, and this contradicts the assumption that he is currently free. ◻

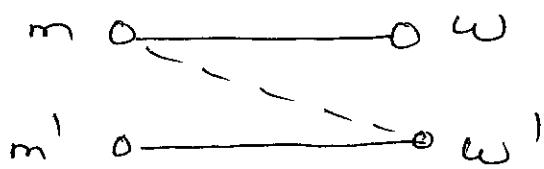
Observation 4: The set  $S$  returned  
returned at termination forms a perfect  
matching.

Proof: The terminating condition of  
the while loop means that at  
termination there is no man who is  
free and has not proposed to ~~some~~ every  
woman. Due to Observation 3, this  
means that there is no man who  
is free (at termination). So  
all men are engaged at termination,  
and the set of engaged pairs  
forms a perfect matching.

Observation 5: The set  $S$  returned by an execution of the G-S algorithm is a stable matching.

Proof: We know that  $S$  is a perfect matching. We will show that there is no instability with respect to  $S$ .

Let  $(m, w) \in M \times W$  be any pair not in  $S$ . So  $(m, w) \in S$  for some ~~w~~  $w \neq w'$ , and  $(m', w') \in S$  for some  $m' \neq m$ .



If  $m$  did not propose to  $w'$ , then we can conclude that  $m$  prefers  $w$  to  $w'$ , because  $m$ 's proposals are ordered by his preference. So there is no danger of  $(m, w')$  being an instability.

If  $m$  did propose to  $w'$ , then since  $m$  is not currently engaged to  $w'$ ,  $w'$  rejected  $\overset{m}{\textcircled{O}}$  (either at the time of  $m$ 's proposal or by later breaking engagement with  $m$ ) in favor of  $m''$  to whom she was engaged.

Either  $m' = m''$ , or by observation 2  $w'$  prefers  $m'$  to  $m''$ .  
In either case, we see that  $w'$  prefers  $m'$  to  $m$ , so  $(m, w')$  can't be an instability.

We conclude that  $S$  is stable □