

Consider the situation where we have a set of  $n$  men and  $n$  women. Each man has a ranking of the  $n$  women, and each woman has a ranking of the  $n$  men.

We'd like to devise a system by which the  $n$  women and  $n$  men can end up getting married - each individual is to be paired with exactly one individual of the opposite gender.

We'd like the set of marriages to be stable. What do we mean by this?

Consider the situation where  $n = 2$ .

$$M = \{m_1, m_2\} \quad W = \{\omega_1, \omega_2\}.$$

Preference lists:

$m_1$  prefers

$\omega_1$  to  $\omega_2$

$m_2$  prefers

$\omega_1$  to  $\omega_2$

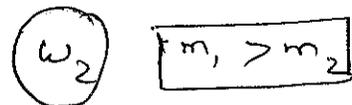
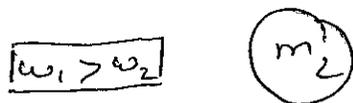
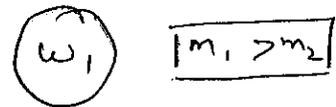
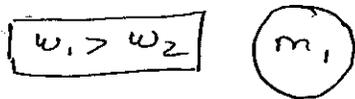
$\omega_1$  prefers

$m_1$  to  $m_2$

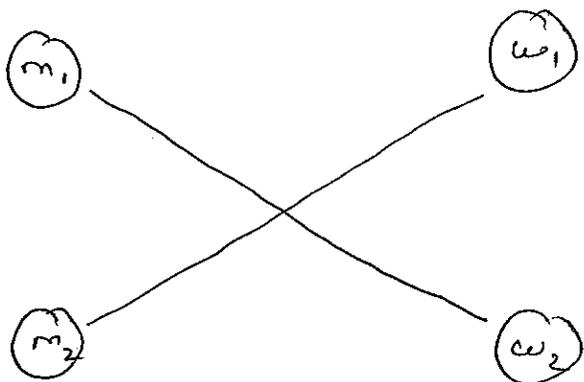
$\omega_2$  prefers

$m_1$  to  $m_2$ .

Pictorially:



Consider the arrangement :



This is unstable, because  $m_1$  and  $w_1$  prefer each other to their current partners.

On the other hand,



is stable.

# The Stable Marriage Problem

We have a set  $M = \{m_1, m_2, \dots, m_n\}$  of  $n$  men, and a set ~~of~~  $W = \{w_1, w_2, \dots, w_n\}$  of  $n$  women.

$M \times W$  denotes set of all ordered pairs of form  $(m, w)$  where  $m \in M$  and  $w \in W$ .

A matching  $S$  is a subset of  $M \times W$  such that each member of  $M \cup W$  appears in at most one pair in  $S$ . A perfect matching is a matching with property that each member of  $M \cup W$  appears in exactly one pair in  $S$ .

Each individual's ranking is given by a preference list.

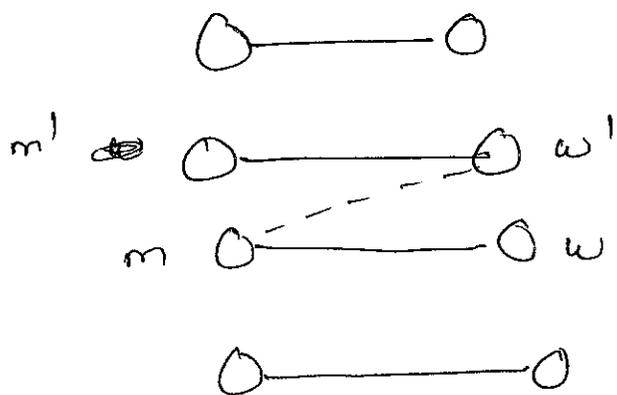
An instability with respect to a perfect matching  $S$  is a pair

$(m, w')$  such that

(1)  $(m, w') \notin S$ .

~~(2)  $m$  prefers  $w'$  to the partner of  $w'$  in  $S$ .~~

(2)  $m$  prefers  $w'$  to his partner in  $S$ , and  $w'$  prefers  $m$  to her partner in  $S$ .



A matching  $S$  is stable if

- (1) it is perfect
- (2) there is no instability with respect to  $S$ .

Questions :

(a) Does a stable matching exist in any situation?

~~(b)~~ Note that this question is not an algorithmic one.

(b) Can we efficiently construct a stable matching if there is one?

This is an algorithmic question.

Knowing that there is a stable

matching, the obvious way of finding one iterates through each of the perfect matchings, and checks if the matching is stable.

No. of perfect matchings is

$$n! \equiv n \times (n-1) \times \dots \times 2 \times 1.$$

( $n$  choices for  $m_1$ ,  $n-1$  choices for  $m_2$  after  $m_1$  has made his choice,  $n-2$  choices for  $m_3$ , ..., choice for  $m_n$ )

A method that takes  $n!$  computational steps will run for several hours on our laptops even for  $n = 12, \dots, 15$ .

Can we do better than brute force search?

Lets consider one more situation.

$$\omega_1 > \omega_2 \quad (m_1)$$

$$(\omega_1) \quad m_2 > m_1$$

$$\omega_2 > \omega_1 \quad (m_2)$$

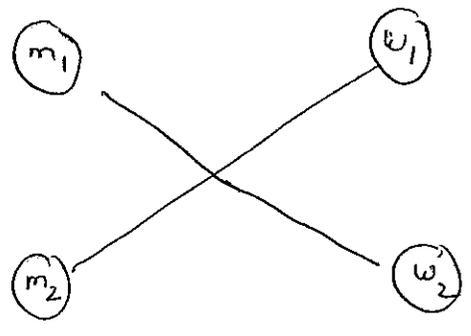
$$(\omega_2) \quad m_1 > m_2$$

The arrangement



is stable.

So is



To solve (a), let us come up with a method that finds a stable matching.

### Attempt 1

$m_1$  chooses a woman he likes best. He is married to the woman he likes best.

$m_2$  is then married to the woman he likes best among the remaining women.

And so on.

This may not work - why should there not be an instability involving  $m_1$ 's partner?

## Attempt 2

If there is a match made in heaven, i.e., a pair  $(m, w)$  so that  $m$  prefers  $w$  to everyone else,  $w$  prefers  $m$  to every one else, choose this pair.

Recurse on the remaining men and women.

Problem: There may not be a match made in heaven, as in our second example.

## Attempt 3:

Start with a perfect matching  $S$ .

~~Repeat till there is no instability~~

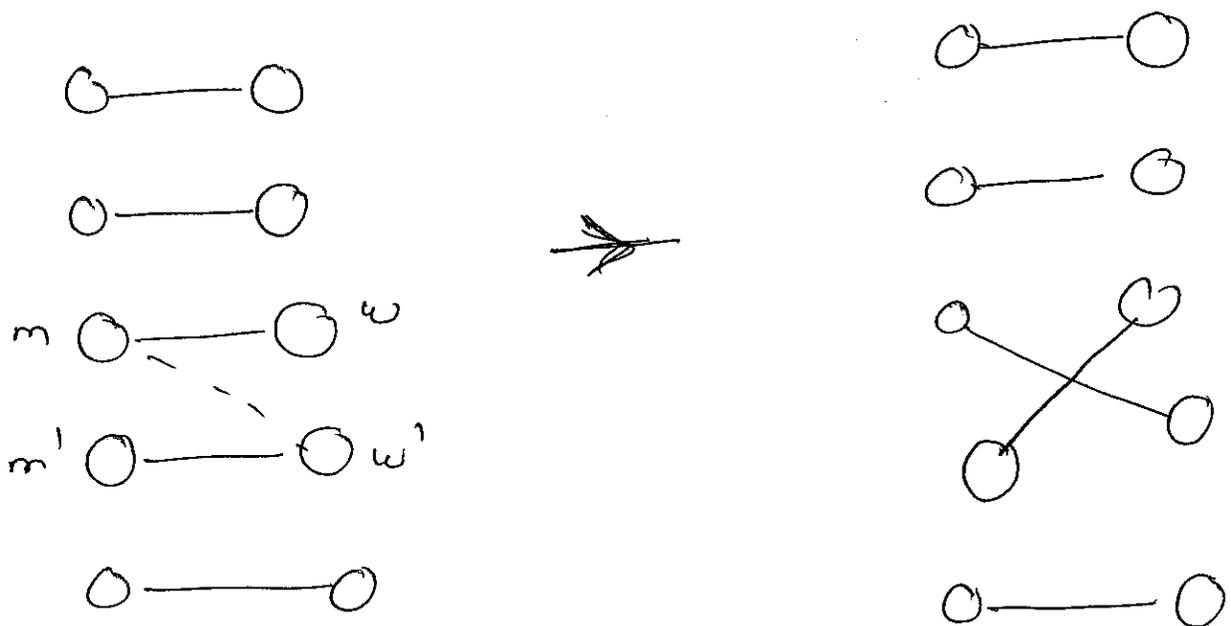
while there is an instability  
 $(m, w')$  with respect to  $S$

- Suppose  $(m, w), (m', w') \in S$ .

- Remove these two pairs from  
 $S$  and replace with  $(m, w')$   
and  $(m', w)$ .

endwhile

Return  $S$



Problem: Does this terminate?  
Why?