Duality principles in analysis

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Several versions of spectral duality are presented. On the two sides we present (1) a basis condition, with the basis functions indexed by a frequency variable, and giving an orthonormal basis; and (2) a geometric notion which takes the form of a tiling, or a Iterated Function System (IFS). Our initial motivation derives from the Fuglede conjecture, see [3, 6, 7]: For a subset D of $\mathbb{R}^n$ of finite positive measure, the Hilbert space $L^2(D)$ admits an orthonormal basis of complex exponentials, i.e., D admits a Fourier basis with some frequencies L from $\mathbb{R}^n$, if and only if D tiles $\mathbb{R}^n$ (in the measurable category) where the tiling uses only a set T of vectors in $\mathbb{R}^n$. If some D has a Fourier basis indexed by a set L, we say that (D, L) is a spectral pair. We recall from [9] that if D is an n-cube, then the sets L in (1) are precisely the sets T in (2). This begins with work of Jorgensen and Steen Pedersen [9] where the admissible sets $L = T$ are characterized. Later it was shown, [5] and [10] that the identity $T = L$ holds for all n. The proofs are based on general Fourier duality, but they do not reveal the nature of this common set $L = T$. A complete list is known only for $n = 1$, 2, and 3, see [9].

We then turn to the scaling IFS's built from the n-cube with a given expansive integral matrix A. Each A gives rise to a fractal in the small, and a dual discrete iteration in the large. In a different paper [8], Jorgensen and Pedersen characterize those IFS fractal limits which admit Fourier duality. The surprise is that there is a rich class of fractals that do have Fourier duality, but the middle third Cantor set does not. We say that an affine IFS, built on affine maps in $\mathbb{R}^n$ defined by a given expansive integral matrix A and a finite set of translation vectors, admits Fourier duality if the set of points L, arising from the iteration of the A-affine maps in the large, forms an orthonormal Fourier basis (ONB) for the corresponding fractal μ in the small, i.e., for the iteration limit built using the inverse contractive maps, i.e., iterations of the dual affine system on the inverse matrix A^{-1}. By "fractal in the small", we mean the Hutchinson measure μ and its compact support, see [4]. (The best known example of this is the middle-third Cantor set, and the measure μ whose distribution function is corresponding Devil's staircase.)

In other words, the condition is that the complex exponentials indexed by L form an ONB for $L^2(\mu)$. Such duality systems are indexed by complex Hadamard matrices H, see [9] and [8]; and the duality issue is connected to the spectral theory of an associated Ruelle transfer operator, see [1]. These matrices H are the same Hadamard matrices which index a certain family of quasiperiodic spectral pairs (D, L) studied in [6] and [7]. They also are used in a recent construction of Terence Tao [11] of a Euclidean spectral pair (D, L) in $\mathbb{R}^5$ for which D does not a tile $\mathbb{R}^5$ with any set of translation vectors T in $\mathbb{R}^5$.

We finally report on joint research with Dorin Dutkay where we show that all the affine IFS's admit wavelet orthonormal bases [2] now involving both the $\mathbb{Z}^n$ translations and the A-scalings.

References

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