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Analysis and Probability

Wavelets, Signals, Fractals

with graphics by Brian Treadway
51 figures and illustrations

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Dedicated to the memory of Shizuo Kakutani

About the cover figure

The figure in the lower right-hand corner of the book cover illustrates the three themes from inside the book, *wavelets, signals, and fractals*: Firstly, this graph is one of the functions which are generated with the use of four magic numbers. Specifically, these same numbers also start an algorithmic generation of Daubechies wavelet functions, and the reader is referred to Figures 7.5 and 7.16 on pages 120–121 and 134 for more images, for mathematical background and explanations, more figures, as well as for theory and exercises in Chapter 7 itself.

To the experts: Figure 7.5 includes the cover figure, and it represents the progression of functions which starts with one of the four-tap cases, and which is governed by the so-called pyramid algorithm. The pyramid algorithm in turn is a delicate design used in the generation of sequences of basis functions. These functions are then further scaled and used in the representation of *wavelet packets*. And the representations and the choices leads to new bases, selections from "libraries of bases" and based on entropy considerations; hence probability!

More probability: The random-walk approach to the analysis of pyramid algorithms in turn is where the calculus of probability comes into the mix. And signals: To begin with, the use of four magic numbers is by adaptation from an algorithmic design which was first used by engineers, and which is fundamental in signal processing; now more recently adapted to image processing as well. But the particular function on the cover also represents the kind of sound signals that feature a beat. A quick glimpse of the figure finally reveals its fractal nature: By this we mean that shapes in a picture are repeated at different scales up to similarity; and which further display an underlying algorithm. Example: Large-scale shapes which envelop similar shapes at smaller scales!

Preface

> *If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.* —John von Neumann

While this is a course in analysis, our approach departs from the beaten path in some ways. Firstly, we emphasize a variety of connections to themes from neighboring fields, such as wavelets, fractals and signals; topics typically not included in a graduate analysis course. This in turn entails excursions into domains with a probabilistic flavor. Yet the diverse parts of the book follow a common underlying thread, and together they constitute a good blend; each part in the mix naturally complements the other.

In fact, there are now good reasons for taking a wider view of analysis, for example the fact that several applied trends have come to interact in new and exciting ways with traditional mathematical analysis—as it was taught in graduate classes for generations. One consequence of these impulses from "outside" is that conventional boundaries between core disciplines in mathematics have become more blurred.

Fortunately this branching out does not mean that students will need to start out with any different or additional prerequisites. In fact, the ideas involved in this book are intuitive, natural, many of them visual, and geometric. The required background is quite minimal and it does not go beyond what is typically required in most graduate programs.

We believe that now is a good time to slightly widen the horizons of the subject "analysis" as we teach it by stressing its relations to neighboring fields; in fact we believe that analysis is thereby enriched.

Despite the inclusion of themes from probability and even from engineering, the course still has an underlying core theme: A constructive approach to building bases in function spaces. The word "constructive" here refers to our use of recursive algorithms. As it turns out, the algorithmic ideas involved are commonly used in such diverse areas as wavelets, fractals, signal and image processing. And yet they share an underlying analysis core which we hope to bring to light.

Our inclusion here of some applied topics (bordering probability theory and engineering) we believe is not only useful in itself, but more importantly, core mathematics, and analysis in particular have benefited from their many interconnections to trends and influences from the "outside" world.

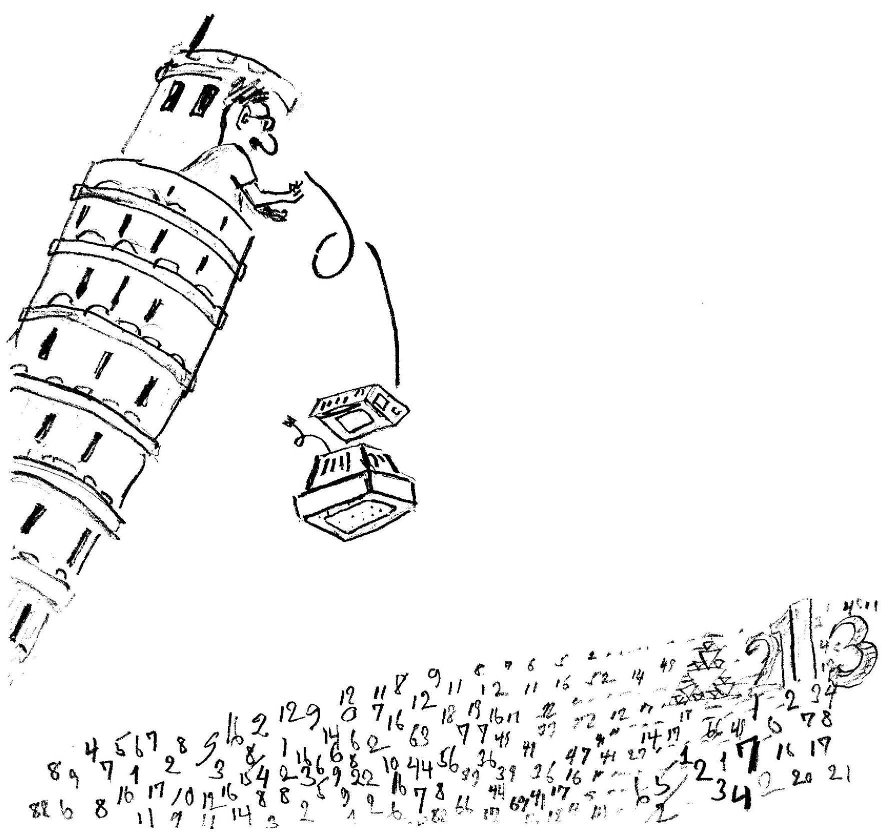
Yet our wider view of the topic analysis only entails a minor adjustment in course planning. Our branching out to some applications will be guided tours: to topics from probability theory (e.g., to certain random-walk models), and to signal and image processing. The ideas are presented from scratch, are easy to follow, and they do not require prior knowledge of probability or of engineering. But we will go a little beyond the more traditional dose of measure theory and matrix algebra that is otherwise standard or conventional fare in most first-year graduate courses.

For those reasons we believe the book may also be suitable for a "second analysis course," and that it leaves the instructor a variety of good options for covering a selection of neighboring disciplines and applications in more depth.

Iowa City,
June 2006 *Palle E. T. Jorgensen*

Contents

Drawing by the author, next page:
Wavelet algorithms are good for vast sets of numbers.
An engineering friend described the old approach to data mining as
"Just drop a computer down onto a gigantic set of unstructured numbers!"
*(*data mining: *see Section 6.2, pp. 102–105, and the Glossary, pp. xxiv–xxv).*



Getting started

From its shady beginnings devising gambling strategies and counting corpses in medieval London, probability theory and statistical inference now emerge as better foundations for scientific models, especially those of the process of thinking and as essential ingredients of theoretical mathematics, even the foundations of mathematics itself. —David Mumford

An apology

You ask: "Why all the fuss?" — Wavelets, signals, fractals? Isn't all of this merely a fad? Or a transient popularity trend? And what's the *probability* part in the book all about? And non-commuting operators? As for bases in linear spaces, what's wrong with Gram–Schmidt?

You may think: "Fourier has served us well for ages; so why do we need all the other basis functions?" — Wavelets and so on? — And why engineering topics in a mathematics course? And the pictures? Are they really necessary?

And there are signal processing and image processing!? — Yes, technology is lovely, but why not leave it to the engineers?

Response: The links between mathematics and engineering are much deeper than the fact that we mathematicians teach service courses for engineers. Our bread and butter!

Mathematics draws ideas and strengths from the outside world, and the connections to parts of engineering have been a boon to mathematics: From signal processing to wavelet analysis! That is true even if we forget about all of the practical applications emerging from these connections. Without inspiration from the neighboring sciences, mathematics would in all likelihood become rather sterile, and overly formal. I see opportunities at crossroads. In this book you will see the benefits mathematics is reaping from trends and topics in engineering. It is witnessed in a striking way by exciting developments in wavelets. From wavelets we see how notions of scale-similarity can be exploited in basis computations that use tricks devised for signal processing. Just open the book and glance at some of the wavelet functions. At the same time, the key notion of self-similarity, such as the scale-similarity used everywhere for wavelets, is essential to our understanding of fractals: Fern-like pictures that look the same at small and at large scales. One problem in the generation of wavelet bases is selecting the “nice” (here this means differentiable) wavelets among huge families of fractal-looking (non-smooth, or singular) functions. L^2-functions can be very “bad” indeed!! Computers generate the good and the bad, and we are left with the task of sorting them out and making selections. We will see (directly from large libraries of pictures) that mathematical wavelet machines are more likely to spit out bad functions unless they are told where to concentrate the search from the intrinsic mathematics.

These wavelets, signals, and fractals are things that have caught our attention in recent decades, but the mathematical part of this has roots back at least a hundred years, for example, to Alfred Haar and to Oliver Heaviside at the turn of the last century. From Haar we have the first wavelet basis, and with Heaviside we see the beginning of signal analysis. It is unlikely that either one knew about the other. Ironically, at the time (1909), Haar’s paper had little impact and was hardly noticed, even on the small scale of “notice” that is usually applied to mathematics papers. Haar’s wonderful wavelet only began to draw attention in the mid-nineteen-eighties when the connections to modern signal processing became much better understood. These connections certainly served as a main catalyst in what are now known as wavelet tools in pure and applied mathematics. But at the outset, the pioneers in wavelets had to “rediscover” a lot of stuff from signal processing: frequency bands, high-pass, low-pass, analysis and synthesis using down-sampling, and up-sampling, reconstruction of signals, resolution of images; all tools that have wonderful graphics representations in the engineering literature.

But still, why would we think that Fourier’s basis, and his lovely integral decomposition, are not good enough? Many reasons: Fourier’s method has computational drawbacks. This was less evident before computers became common and began to play important roles in applied and theoretical work. But expansion of functions or signals into basis decompositions (called “analysis” in signal processing) involves basis coefficients (Fourier coefficients, and so on), and if we are limited to Fourier bases, then the computation of the coefficients must by necessity rely on integration. “Computers can’t integrate!” Hmmm! Well, not directly. The problem must first be discretized. And there is need for a more direct and algorithmic approach. Hence the wavelet algorithm! In any case, algorithms are central in mathematics even if you do not concern yourself with computers. And it is the engineering connections that inspired the most successful algorithms in our subject.