The Pathologies of Voting Schemes

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1 Introduction

In 2016, Trump pulled a shocking upset, winning 306 electoral votes and the presidency, despite losing the popular vote by almost 3 million votes [7]. This phenomenon, where the popular vote winner loses the election, is known as an electoral inversion and has occurred 4 times over the history of US elections. A study by Geruso, Spears, and Talesara found that probability of an electoral inversion occurring is incredibly high in close elections. In elections with a 2% popular vote margin, the probability of an inversion is more than 30%, while in elections with a 1% margin, the probability rises to over 40%. Given that 1 in 8 elections since 1828 have had a popular vote margin within 1%, it is highly likely that voters will experience at least one inversion within their lifetime [4].

One of the factors that results in such a high chance for inversions is the winner-take-all nature of the Electoral College. In 48 of the 50 states, all of the states electors are awarded to whomever gets the most votes in the state [2]. While this scheme, also known as First Past the Post, seems simple and intuitive, it leads to a number of problems. In this paper, we explore the theory of voting schemes and consider alternatives to First Past the Post.

2 Theory of Voting

At its core, voting is a vehicle for individuals to express their preferences, with the goal of aggregating the results to produce a single group ranking [3]. To formalize this, we define a preference relation as follows. Given an individual i and two choices x and y, we can express their opinion on these two choices as $x \succeq_i y$ if they prefer x at least as much as y. Note that this choice of a preference relation allows a voter to be indifferent between two choices. If they are indifferent, then both $x \succeq_i y$ and $y \succeq_i x$ are true.

There are some properties we assume about the preference relation, namely completeness and transitivity. Both are relatively intuitive. Completeness states that for any two choices x and y in the domain, we either have $x \succeq_i y$ or $y \succeq_i x$. Transitivity states that if we have $x \succeq_i y$ and $y \succeq_i z$, then we also have $x \succeq_i z$. While non-transitive preferences can certainly exist, we ignore them for simplicity. One consequence of accepting these two properties is that we can show the preference relation defines a total ordering. This means that the preference relation can be viewed as a ranked list of choices [3]. Hereafter, we use these two notions interchangeably.

Given a group of individual preferences, a voting scheme is any method that takes these relations and produces a single ranking representing the opinion of the entire group [3]. Note that this definition does not provide any built-in notion of fairness. Randomly picking an individual's ranking to represent the group would be a valid voting scheme, albeit a terrible one.

There are many possible criteria for fairness. While theorist don't agree on the all the criteria, they generally agree on a minimal set: anonymity, neutrality, and unanimity [9]. These properties are defined as follows:

- Anonymity - permuting the ballots does not change the outcome
- Neutrality - permuting the order of candidates on the ballot does not change the outcome
- Unanimity - if all individuals agree that $x \succeq y$, then in the group ranking $x \succeq y$

2.1 Arrow's Impossibility Theorem

In 1950, Arrow proved that there are no ranked voting schemes with unanimity, independence of irrelevant alternatives (IIA), and non-dictatorship simultaneously [1]. We have already defined unanimity above. Independence of irrelevant alternatives means that changes in individual preferences of other alternatives should not change the group preference. For instance, if in the group ranking we have $x \succeq y$, then adding or removing some choice z should not affect this preference. Non-dictatorship means that there is no individual whose individual preferences fully determine the group preferences.

While non-dictatorship is obviously a property of a voting scheme we want, the importance of IIA is less clear. IIA may sound like a desirable property, but it is not always obeyed in the real world. The red-bus/blue-bus problem illustrates a situation where IIA does not hold. Suppose that a group of commuters travel to work either by car or by red buses, and that 40% of commuters travel by car and the other 60% travel by red bus. This means in the group ranking we have $redbus \succeq car$. Now suppose we add a third method of transportation, blue buses and further assume that commuters don't particularly care about the color of their bus. Intuitively, we would expect some of the red bus commuters to take the blue bus instead, so that 40% of commuters travel by car, 30% travel by red bus, and 30% travel by blue bus. However, in our group ranking, we have $car \succeq redbus$, violating IIA.

2.2 Gibbard–Satterthwaite Theorem

As a follow-up to Arrow's Theorem, Gibbard and Satterthwaite independently proved that it is impossible to eliminate strategic voting, where voters are better off voting for someone who isn't their true preference [9]. An example of this would be in the US 2000 elections, where a strategic voter whose preferences are $Nader \succeq Gore \succeq Bush$ may choose to vote for Gore instead to prevent Bush from winning.

2.3 On Two Candidate Elections

It is interesting to note that both Arrow's Theorem and the Gibbard–Satterthwaite Theorem only apply to elections with three or more candidates. In an election with only two candidates, the simple majority rule suffices and behaves as expected. However, in practice, most elections have have more than two choices. Even in the United States, where there are two predominant major parties, there were 224 state-level ballot-qualified political party affiliates as of September 2019 [6]. In some states, there are as many as ten political parties on the ballot.

3 Common Voting Pathologies

From the previous section, we know that it is impossible to find a perfect voting schemes free of pathological behavior. However, it does not imply that all voting schemes are equally bad, as the effects of different pathological behaviors are not the same. Let us examine some common issues that arise with various voting schemes.

3.1 First Past the Post (FPTP) and Lesser Evil

As the simple majority rule works for elections with two candidates, it is tempting to generalize it to elections with more candidates. In FPTP, each voter casts a single vote for a candidate and the candidate with the most votes is selected. This system is used in a quarter of the world's countries, including the United States, the United Kingdom, and Canada [14].

The primary problem is that voters only have a single vote, so voting for a fringe third-party candidate who eventually loses is effectively the same as not voting. Thus, a strategic voter is incentivized to vote for the least-bad of the viable options to avoid "wasting their vote" [8]. The logical result of this is that eventually, there will be only two primary parties. This observation is known as Duverger's Law.

3.2 Borda Count and Dark Horse

One way to fix the issue of having only a single vote is to give voters multiple votes. A simple system that does this is Borda Count. In an election with n candidates, each voter ranks the candidates according to their preferences. Points are given to candidates based on their position in each individual ranking, so a candidate receives n points if they are ranked first, $n - 1$ points for being ranked second, and so on.

However, Borda Count reacts poorly to strategic voting. In order to maximize the chances of one's preferred candidate winning, it is optimal to rank the strongest competitors at bottom to reduce the number of points they get. This means that weak candidates will be ranked in the middle, as everyone has to be ranked under Borda Count. However, if everyone does this, the weak candidates will gain a majority of the points and win the election, despite not being truly supported by many people.

3.3 Instant Run-off Voting (IRV) and Center Squeeze

A different approach to fixing FPTP is minimize the number of wasted voting going to minor candidates. Under IRV, voters choose any number of candidates to vote for and rank them in the order they support them in. If there is a candidate with over 50% of the votes, they win the election. If there isn't, the candidate with the least support is eliminated and the votes which went to the candidate are given to the voters' second choice. This process is repeated until one candidate has the majority.

The problem with IRV is a bit more subtle than the previous ones. Consider a situation with three candidates A, B, C and where 40% of the voters prefer $A \succeq B \succeq C$, 40% of the voters prefer $C \succeq B \succeq A$, 10% prefer $B \succeq A \succeq C$, and the remaining 10% prefer $B \succeq C \succeq A$. Who should win this election?

If we look at the pairwise comparisons, we see that 60% of the voters prefer B over A and 60% prefer B over C, suggesting that candidate B has the greatest overall support. However, under IRV, candidate B is eliminated in the first round. This phenomenon is referred to as the center squeeze as B is a centrist who is the second choice, but not a first choice for many people, and thus gets "squeezed out" in the first round.

Despite this issue, IRV is still an improvement over FPTP, as the problem only arises when candidates receives between 25% and 50% of the total support, rather than immediately disincentivizing third-party candidates from running.

3.4 Approval Voting and Chicken Dilemma

Approval voting is a simpler system than IRV. Instead of ranking the candidates, the voter simply votes for as many candidates as they wish. Determining the winner is also simpler, as it is just the winner with the most votes.

Consider the following situation with three candidates where 30% of the voters prefer $A \succeq B$, 30% prefer $B \succeq A$, and 40% prefer C. Under approval voting, A and B are tied for first place. However, if some of the voters for A strategically vote only for A, then A becomes the winner. Similarly voters for B are incentivized to vote solely for B. The problem is if both voters for A and B vote for their own candidates, then C wins the election, despite not having a majority of the support.

3.5 Condorcet Cycles

Another way to determine the winner in an election is to look at all the contests between every pair of candidates. The candidate that wins all the pairwise contests is known as the Condorcet winner. We used this method when discussing IRV to determine that B had the greatest overall support. Condorcet methods refer to a family of voting schemes that ask voters to rank their preferences and gives the Condorcet winner if it exists.

A Condorcet cycle occurs when there is no candidate that is the pairwise-winner over all opponents [5]. Consider a situation with three voters and three candidates A, B, and

C. One voter prefers $A \succeq B \succeq C$, one voter prefers $B \succeq C \succeq A$ and one voter prefers $C \succeq A \succeq B$. In this situation, there is no Condorcet winner. Every candidate wins one pairwise contest and loses one pairwise contest.

In fact, this situation is an example of why strategic voting is unavoidable under any voting scheme. In this situation, if any voter votes for their second choice, there will be a majority for that choice [8]. Condorcet cycles are inherently unresolvable. Fortunately, they occur relatively rarely in practice.

4 An Example Election

To illustrate how different voting schemes may disagree on the winner of an election, we look at an example election with voters' preferences shown in figure 1. Highlighted in red are the candidates that would by approved under approval voting.

An 11-voter, 6-candidate election (candidates A,B,C,D,E,F)

#voters	Their Vote
4	A>D>E>C>B>F
3	B>E>D>F>C>A
2	C>B>E>D>F>A
1	D>C>F>A>E>B
1	F>C>E>D>B>A

Figure 1: An example election [13]

Under FPTP, A wins the election, having the most first votes.

Under Borda Count, D wins the election as D's Borda score is $4(5) + 3(4) + 2(3) + 1(6) + 1(3) = 47$. D gets $4(5) = 20$ points for coming in second four times, $3(4) = 12$ points for coming in third three times, and so on. The next highest Borda score is from E, with 45 points and the other candidates have lower scores.

Under IRV, C wins the election. In the first round, there is no majority, so E is eliminated as no one voted for them in their top choice. Then, F is eliminated and the one voter's vote goes to C. Then D is eliminated and the voter's vote goes to C. At this point A has 4 votes, B has 3 votes, and C has 4 votes, so B is eliminated and the three votes go to C (as E, D, and F are already gone), so C wins the election, beating A 7 to 4.

Under Condorcet methods, E wins, beating all other candidates in pairwise comparisons. For example, E is preferred over A $3 + 2 + 1 = 6$ times, while A is preferred over $4 + 1 = 5$ times, so E beats A 6 to 5. Similarly, E beats B 6 to 5, E beats C 7 to 4, E beats D 6 to 5, and E beats F 9 to 2.

Under approval voting, F wins the election, as they are approved 7 times.

5 Comparing Various Schemes

One way to determine the quality of a voting scheme is by computing the Bayesian regret, which is the "expected avoidable unhappiness" caused by the voting scheme [11]. This concept is based on the idea that every voter has a personal utility value for every candidate which is a measure of how happy the voter would be if the candidate were elected. At

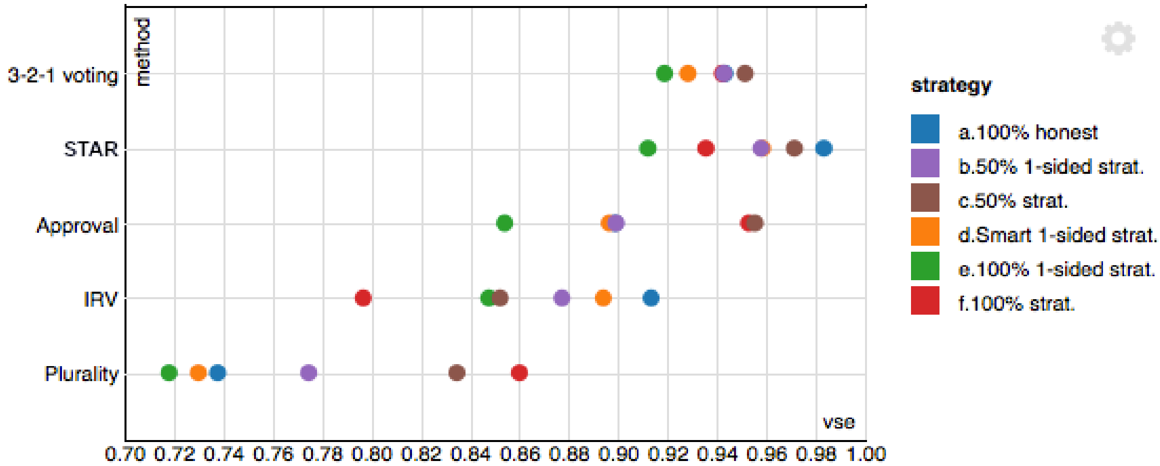

Figure 2: Results of simulating voting methods from Jameson Quinn [10]

the end of the election, the utilities for the elected candidate is summed and compared to the maximum possible utility if we somehow chose the best candidate. The difference between the achieved utility and the maximum potential utility is the Bayesian regret. A related concept is the Voter Satisfaction Efficiency (VSE), which expresses the same idea as Bayesian regret but as a percentage. A method that chooses the candidate maximizing utility receives a VSE of 100%, while a method picking a random candidate has a VSE of 0% [10]. If a method chose the worst possible candidate, the method would have a VSE of -100%. In practice, most methods do better than random choice, with the notable exception of Borda count, which can elect a dark horse winner nobody wanted.

While Bayesian regret is difficult to measure in practice as voters' utilities are hard to quantify and are likely not even on the same scale, it is feasible to compute through computer simulations. To do so, we need to make some assumptions about how voters assign utility, what voters know, and how voters act during the election (whether they are strategic or not) [10]. By repeating the simulation with a variety of parameters, we can get an idea of how a particular method behaves in the best and worst case scenarios.

This process was used by a statistician, Jameson Quinn, to compare various voting schemes and his results are shown in Figure 2. In addition to testing a variety of voting schemes, Quinn also tested a variety of voter strategies to measure how schemes react to strategic voting. The three main strategies tested are:

- Honest: voters pick the candidate they prefer without considering other information
- Strategic: voters find the top two leading candidates and give maximum votes to the candidate they prefer and minimum votes to the other
- One-sided strategy: voters supporting the top candidate vote honestly, but voters supporting the runner-up vote strategically

We can see that FPTP (plurality) gets some of the worst results and changing the voting method to approval or IRV can significantly improve the quality of elections. There are benefits to choosing either of these schemes. Approval voting is a simple reform and most existing ballots can continue to be used, while IRV has already been used in many elections and has the strongest track record [10].

6 Other Schemes: Cardinal Voting

Four of the schemes we have examined (FPTP, Borda count, IRV, and Condorcet) are examples of ordinal voting schemes, where voters rank candidates in order of preference. The other major family of voting schemes are cardinal voting schemes, where voters rate each candidate independently. The two best methods shown in figure 2, STAR and 3-2-1 voting, are examples of cardinal voting schemes.

The simplest, most general cardinal voting scheme is range voting, also known as score voting. In range voting, voters rank candidates on a scale. This is typically from 0 to 5 or 0 to 9, but can be on any scale, depending on the precision desired. Note that approval voting is a special case of range voting, with two possible levels (”approve” or ”disapprove”).

Range voting has a number of advantages over ordinal voting schemes. One of the largest advantages is the increased expressiveness voters have. Under range voting, a voter can express not only that they prefer candidate X over candidate Y, but also quantify how much they prefer each. Range voting is also far more resistant to strategic voting, as it is always better to give the top score to your preferred candidate [12]. Another interesting property of range voting is that it satisfies all three properties given in Arrow's Impossibility Theorem: non-dictatorship, unanimity, and IIA. This occurs because Arrow's theorem only applies to ordinal voting schemes.

7 Conclusion

We have looked at a number of voting schemes and how they behave and potentially fail to behave as expected. However, it is important to remember that the fairness properties of voting schemes are merely part of the picture. There are many other challenges to getting an alternative voting scheme implemented, such as the difficulty of explaining the voting scheme, the difficulty of auditing or recounting, and how these schemes conform to existing laws. There exist a number of advocacy in the United States to adopt a different voting scheme including FairVote, the Center for Election Science, and the Equal Vote Coalition. While the best voting scheme may be up for debate, almost any scheme is superior to FPTP.

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