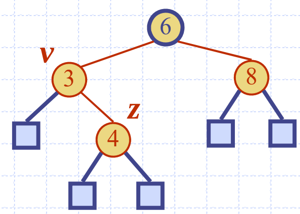
Presentation for use with the textbook Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Ch04 Balanced Search Trees



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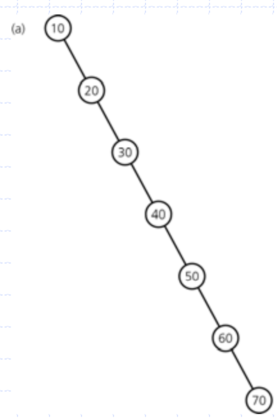
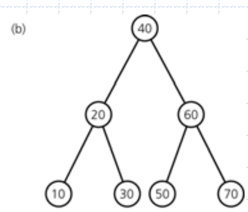
Why care about advanced implementations?

Same entries, different insertion sequence:





→ Not good! Would like to keep tree balanced.

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Balanced binary tree

- The disadvantage of a binary search tree is that its height can be as large as N-1
- This means that the time needed to perform insertion and deletion and many other operations can be O(N) in the worst case
- We want a tree with small height
- A binary tree with N node has height at least $\Theta(\log N)$
- Thus, our goal is to keep the height of a binary search tree O(log N)
- Such trees are called balanced binary search trees. Examples are AVL tree, and red-black tree.

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Approaches to balancing trees

- Don't balance
  - May end up with some nodes very deep
- Strict balance
  - The tree must always be balanced perfectly
- Pretty good balance
  - Only allow a little out of balance
- Adjust on access
  - Self-adjusting

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Balancing Search Trees

- Many algorithms exist for keeping search trees balanced
  - Adelson-Velskii and Landis (AVL) trees (height-balanced trees)
  - Red-black trees (black nodes balanced trees)
  - Splay trees and other self-adjusting trees
  - B-trees and other multiway search trees

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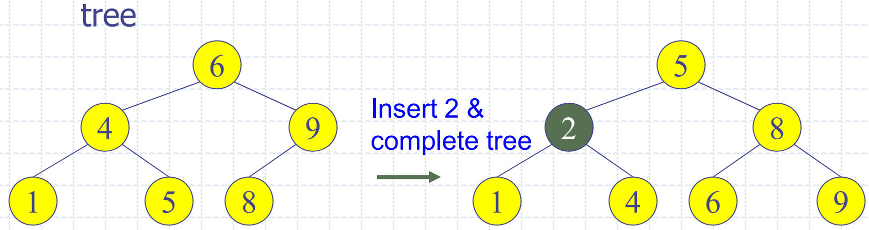
Perfect Balance

- Want a complete tree after every operation
  - Each level of the tree is full except possibly in the bottom right
- This is expensive
  - For example, insert 2 and then rebuild as a complete tree



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AVL - Good but not Perfect Balance

- AVL trees are height-balanced binary search trees
- Balance factor of a node
  - height(left subtree) - height(right subtree)
- An AVL tree has balance factor calculated at every node
  - For every node, heights of left and right subtree can differ by no more than 1
  - Store current heights in each node

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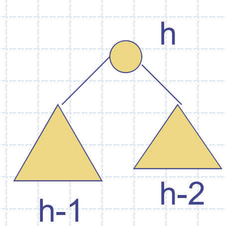
Height of an AVL Tree

- N(h) = minimum number of nodes in an AVL tree of height h.
- Basic case:
  - N(0) = 1, N(1) = 2
- Inductive case:
  - N(h) = N(h-1) + N(h-2) + 1
- Theorem (from Fibonacci analysis)
  - $N(h) \geq \phi^h$

  where $\phi \approx 1.618$, the golden ratio.



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Height of an AVL Tree

- $N(h) \geq \phi^h$ ($\phi \approx 1.618$)
- Suppose we have n nodes in an AVL tree of height h.
  - $n \geq N(h)$ (because N(h) was the minimum)
  - $n \geq \phi^h$ hence $\log_\phi n \geq h$ (relatively well balanced tree!!)
  - $h \leq 1.44 \log_2 n$ (i.e., Find takes O(logn))

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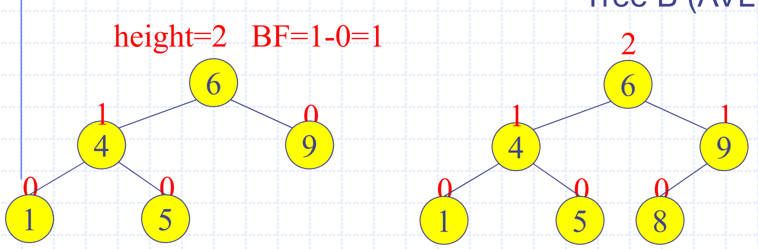
Node Heights

Tree A (AVL) Tree B (AVL)



height of node = h
balance factor = $h_{left}-h_{right}$
empty height = -1

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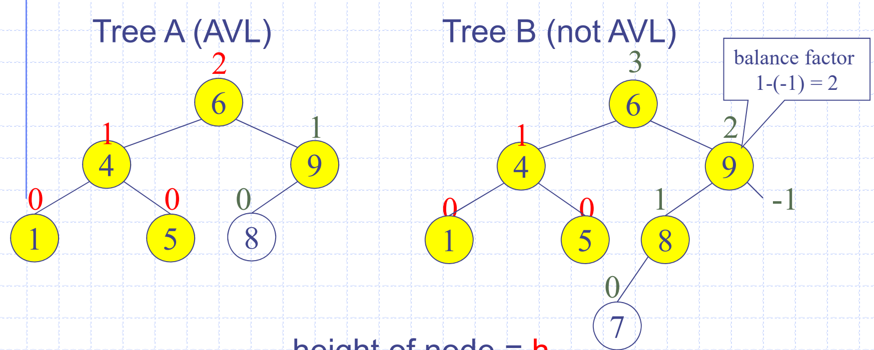
Node Heights after Insert 7

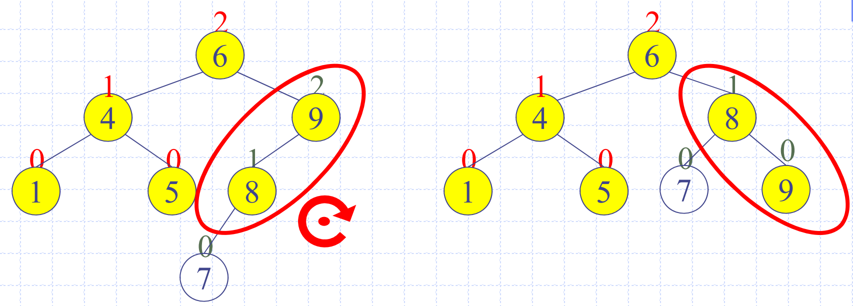


height of node = h
balance factor = h_{left}-h_{right}
empty height = -1

Insert and Rotation in AVL Trees

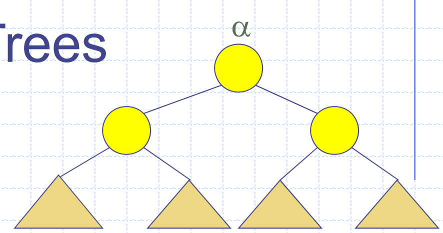
- Insert operation may cause balance factor to become 2 or –2 for some node
  - only nodes on the path from insertion point to root node have possibly changed in height
  - So after the Insert, go back up to the root node by node, updating heights
  - If a new balance factor (the difference h_{left}-h_{right}) is 2 or –2, adjust tree by *rotation* around the node

Single Rotation in an AVL Tree



Insertions in AVL Trees

Let the node that needs rebalancing be α.



Cases: 1 3 4 2

There are 4 cases:

Outside Cases (require single rotation) :

1. Insertion into left subtree of left child of α. (left-left)
2. Insertion into right subtree of right child of α. (right-right)

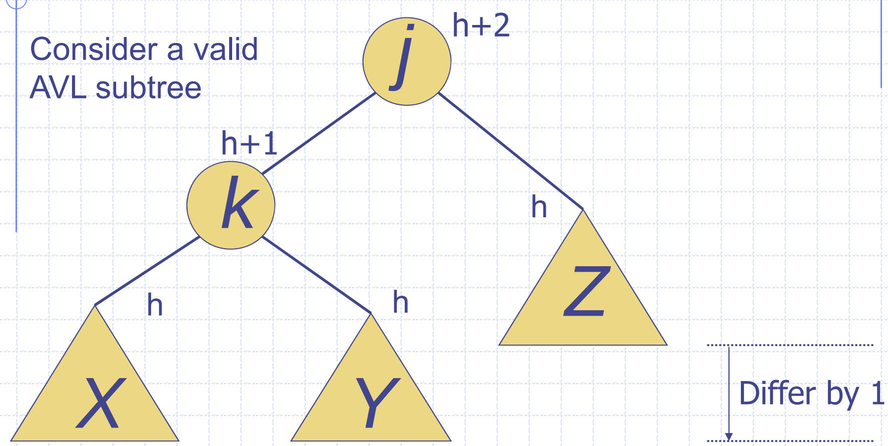
Inside Cases (require double rotation) :

3. Insertion into right subtree of left child of α. (left-right)
4. Insertion into left subtree of right child of α. (right-left)

The rebalancing is performed through four separate rotation algorithms.

AVL Insertion: Outside Case

Consider a valid AVL subtree



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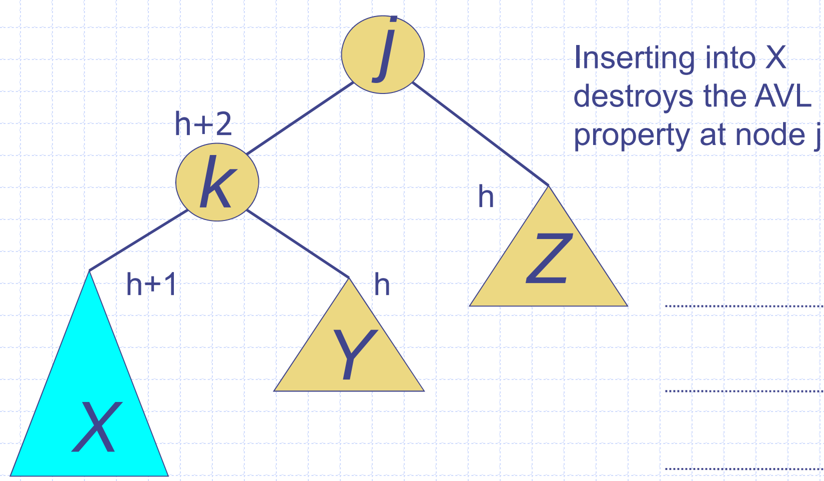
AVL Insertion: Outside Case

Inserting into X destroys the AVL property at node j



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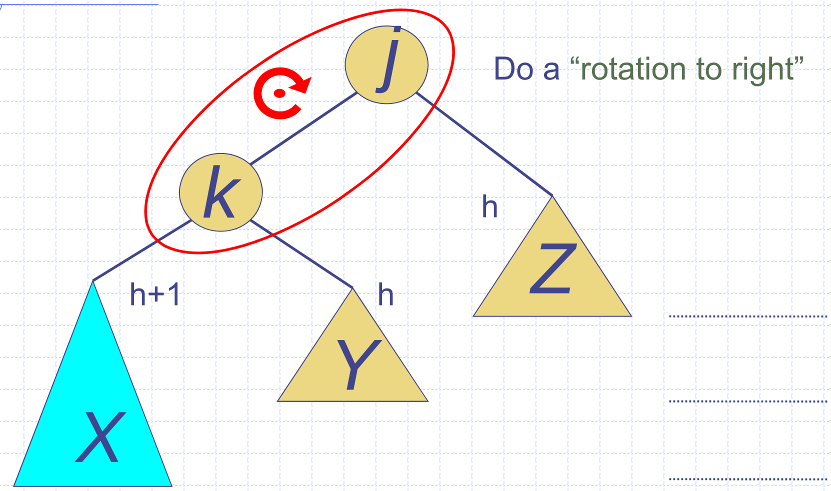
AVL Insertion: Outside Case



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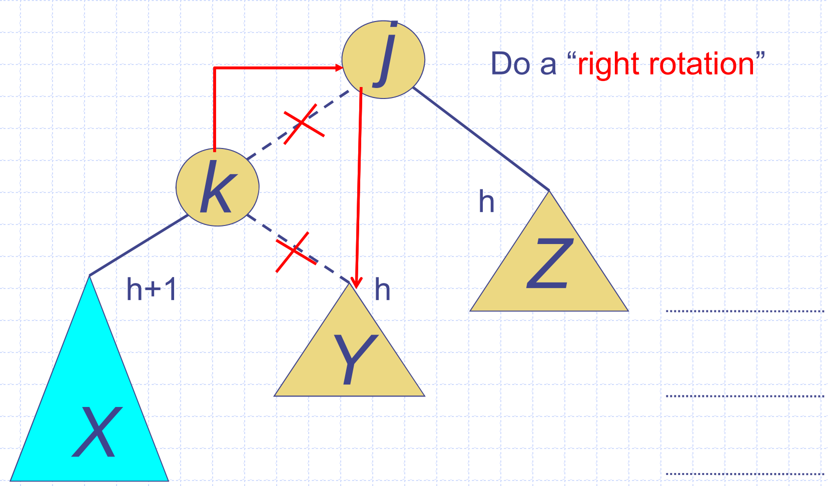
Single right rotation



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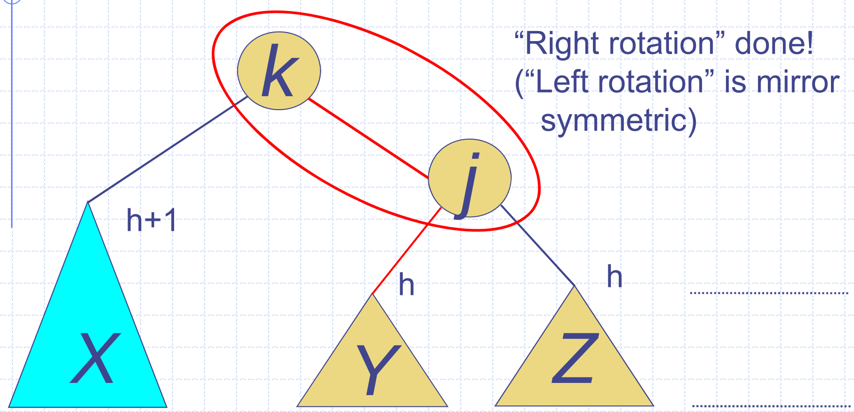
Outside Case Completed

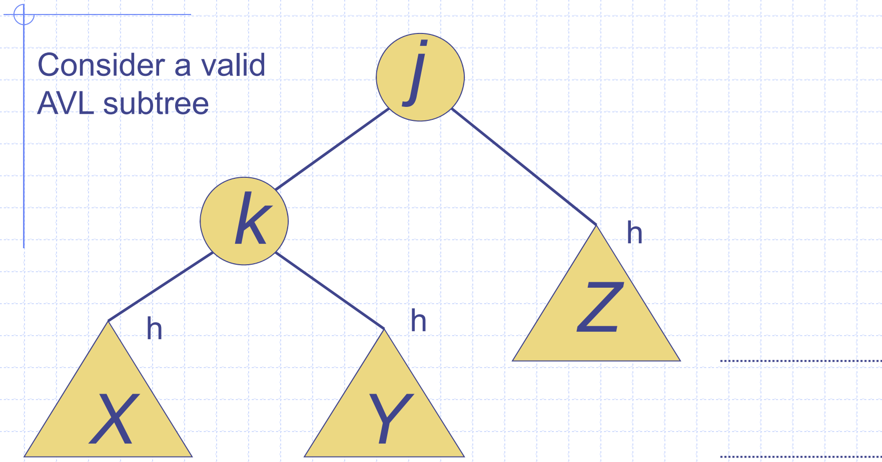
“Right rotation” done!
(“Left rotation” is mirror
symmetric)



AVL property has been restored!

AVL Insertion: Inside Case

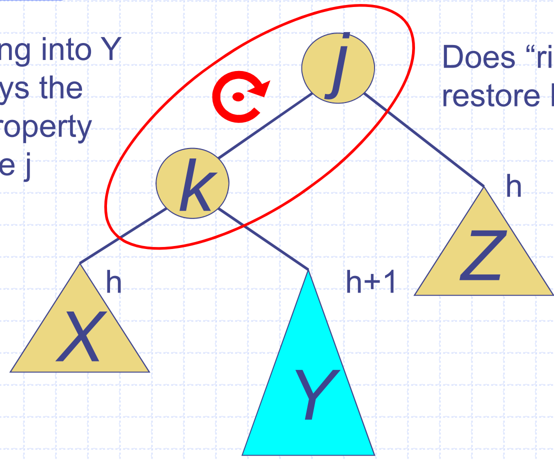
Consider a valid
AVL subtree



AVL Insertion: Inside Case

Inserting into Y destroys the AVL property at node j

Does "right rotation" restore balance?



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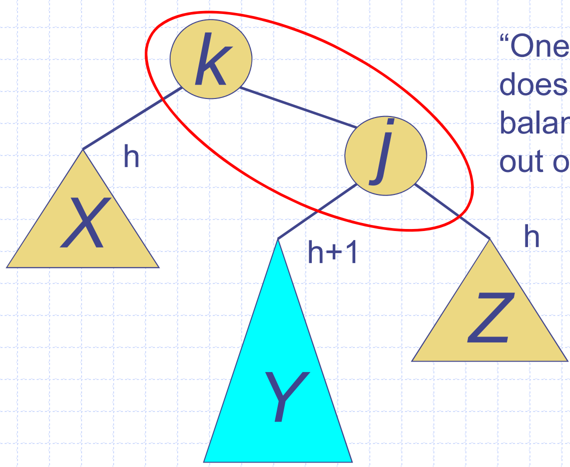
AVL Insertion: Inside Case

"One rotation" does not restore balance… now k is out of balance



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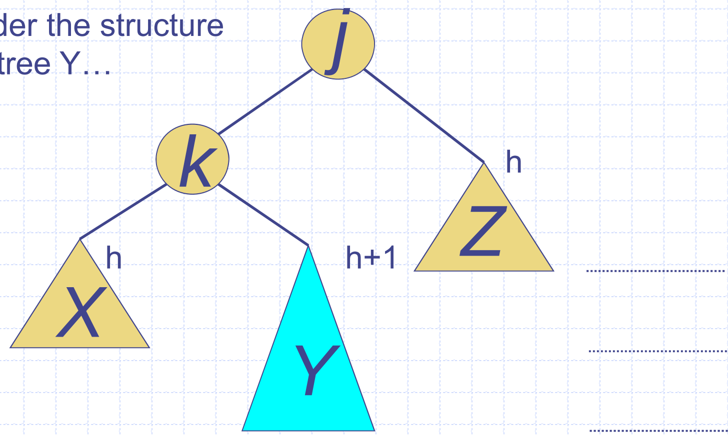
AVL Insertion: Inside Case

Consider the structure of subtree Y…



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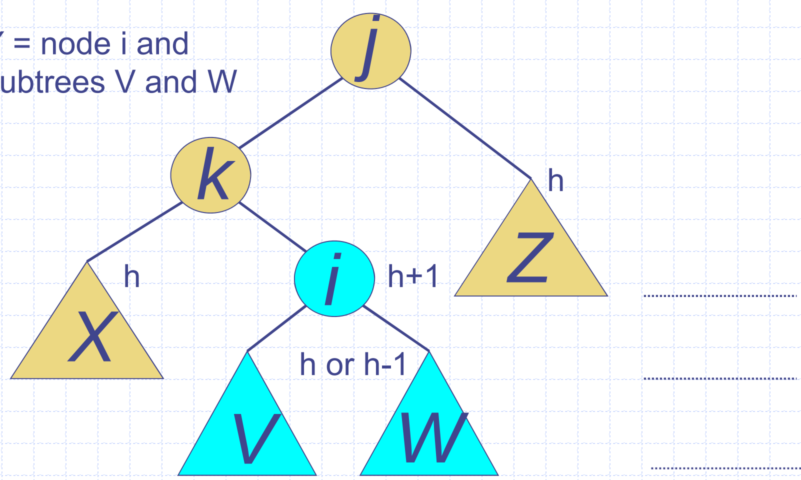
AVL Insertion: Inside Case

Y = node i and subtrees V and W



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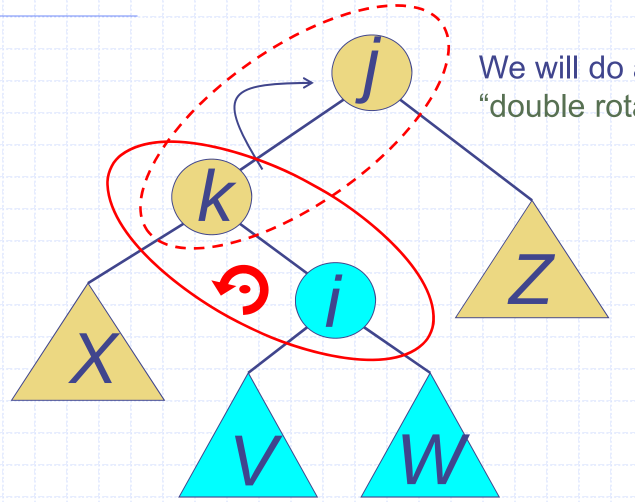
AVL Insertion: Inside Case



We will do a left-right "double rotation" . . .

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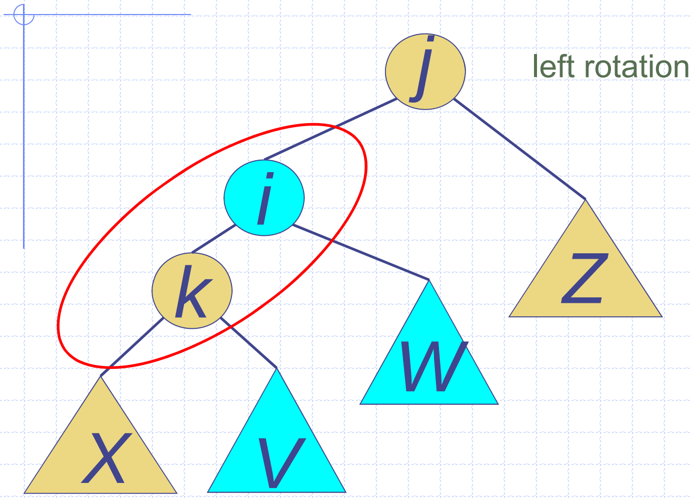
Double rotation : first rotation



left rotation complete

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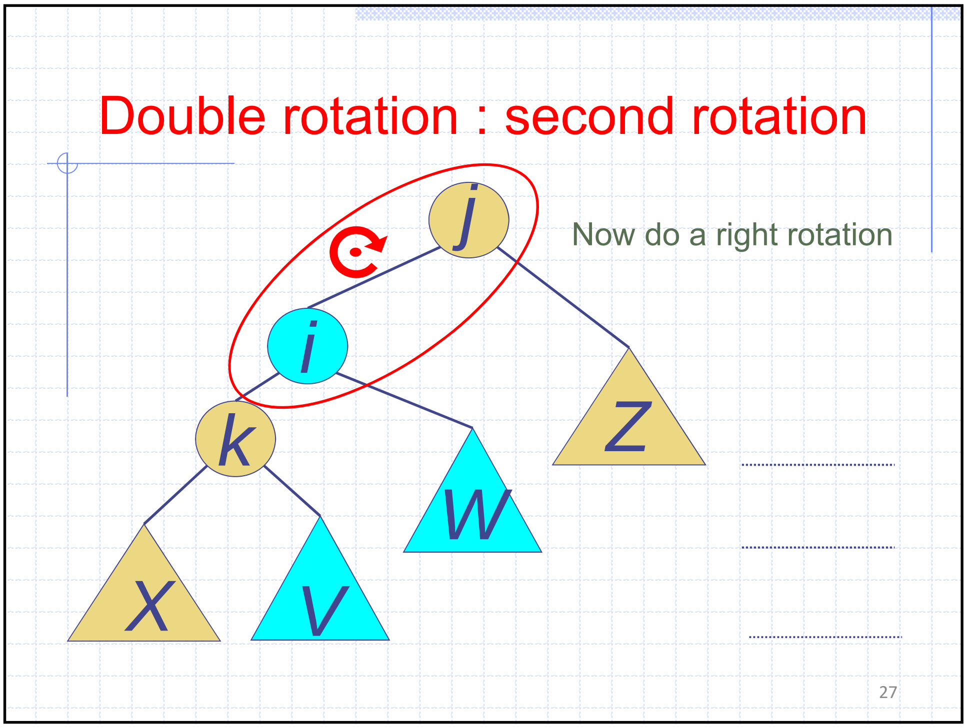
Double rotation : second rotation
Now do a right rotation
j
i
k
Z
W
X
V
27

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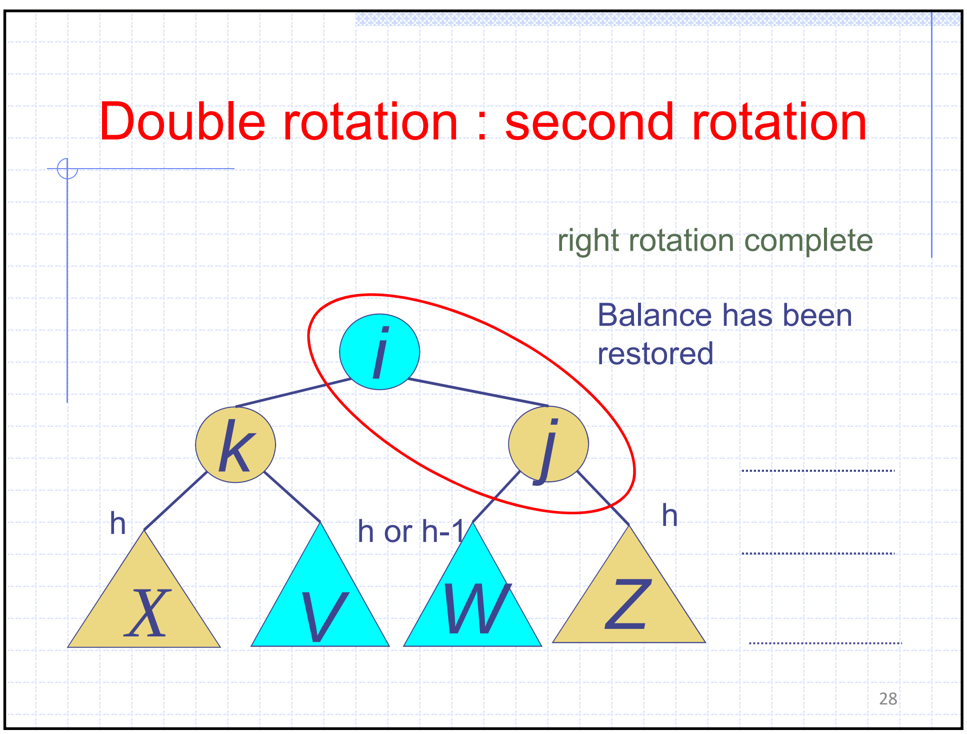
Double rotation : second rotation
right rotation complete
Balance has been restored
i
k
j
h
h or h-1
h
X
V
W
Z
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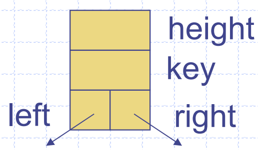
Implementation



Once you have performed a rotation (single or double) you won't need to go back up the tree

```
Class BinaryNode
    KeyType: Key
    int: Height
    BinaryNode: LeftChild
    BinaryNode: RightChild

   Constructor(KeyType: key)
        Key = key
        Height = 0
    End Constructor
End Class
```

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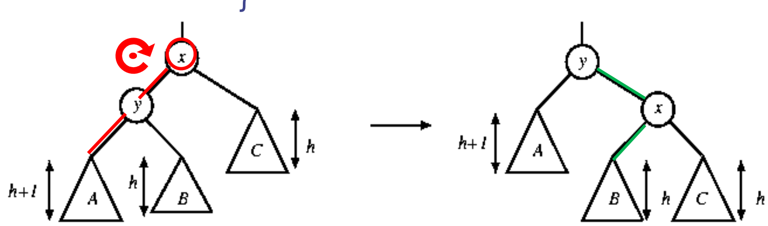
Java-like Pseudo-Code

```
rotateToRight( BinaryNode: x ) {
    BinaryNode y = x.LeftChild;
    x.LeftChild = y.RightChild;
    y.RightChild = x;
    return y;
}
```



Rotate with left child

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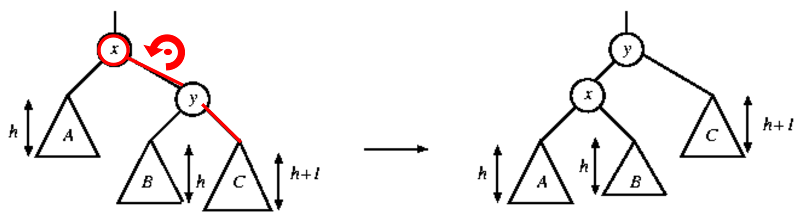
Java-like Pseudo-Code

```
rotateToLeft( BinaryNode: x ) {
    BinaryNode y = x.rightChild;
    x.rightChild = y.leftChild;
    y.leftChild = x;
    return y;
}
```



Rotate with right child

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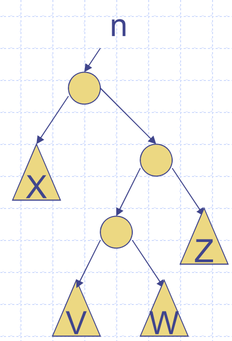
Double Rotation

- Implement Double Rotation in two lines.

```
DoubleRotateToLeft(n : binaryNode) {
  rotateToRight(n.rightChild);
  rotateToLeft(n);
}

DoubleRotateToRight(n : binaryNode) {
  rotateToLeft(n.leftChild);
  rotateToRight(n);
}
```



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Insertion in AVL Trees

- Insert at the leaf (as for all BST)
  - only nodes on the path from insertion point to root node have possibly changed in height
  - So after the Insert, go back up to the root node by node, updating heights
  - If a new balance factor (the difference h_{left}-h_{right}) is 2 or –2, adjust tree by *rotation* around the node

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Insert in ordinary BST

```
Algorithm insert(k, v)
   input: insert key k into the tree rooted by v
   output: the tree root with k adding to v.
   if isNull (v)
       return newNode(k)
   if k ≤ key(v)    // duplicate keys are okay
       leftChild(v) ← insert (k, leftChild(v))
   else if k > key(v)
       rightChild(v) ← insert (k, rightChild(v))
   return v
```

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Insert in AVL trees

```
Insert(v : binaryNode, x : element) : {
  if v = null then
    {v ← new node; v.data ← x; height ← 0;}
  else case
  v.data = x : ; //Duplicate do nothing
  v.data > x : v.leftChild ← Insert(v.leftChild, x);
    // handle left-right and left-left cases
    if ((height(v.leftChild)- height(v.rightChild)) = 2)then
      if (v.leftChild.data > x ) then //outside case
        v = RotateToRight (v);
      else //inside case
        v = DoubleRotateToRightt (v);}
  v.data < x :  v.righChild ← Insert(v.rightChild, x);
    // handle right-right and right-left cases
    … …
  Endcase
  v.height ← max(height(v.left),height(v.right)) +1;
  return v;
}
```

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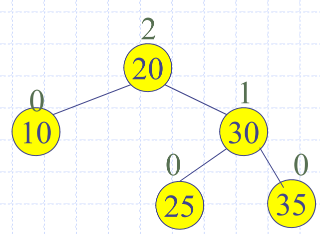
Example of Insertions in an AVL Tree



Insert 5, 40

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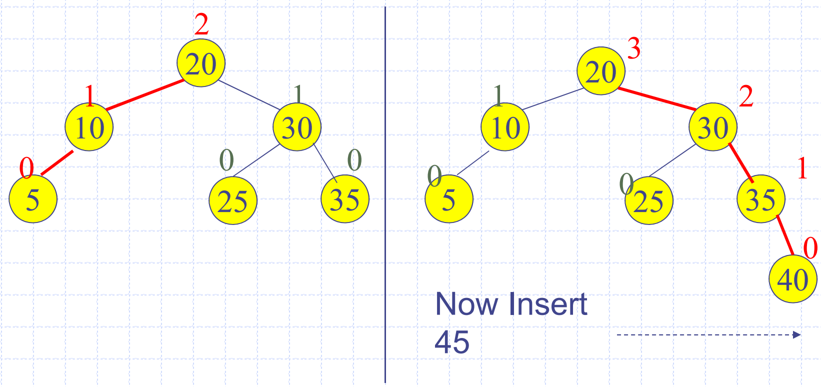
Example of Insertions in an AVL Tree



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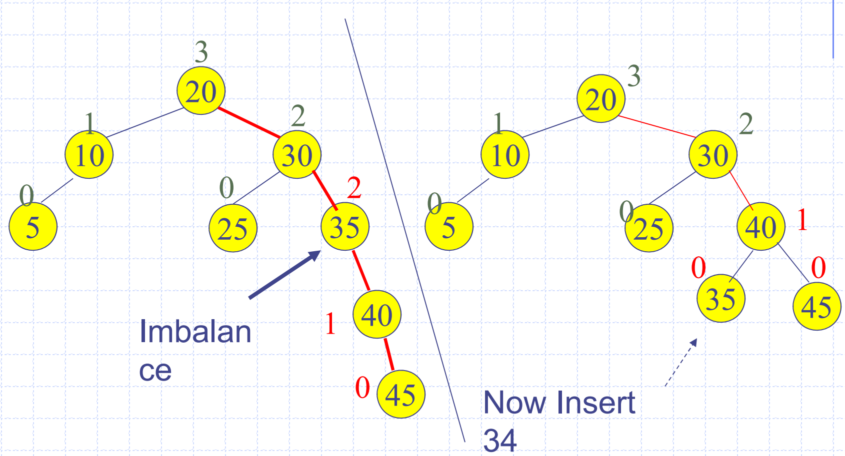
Single rotation (outside case)



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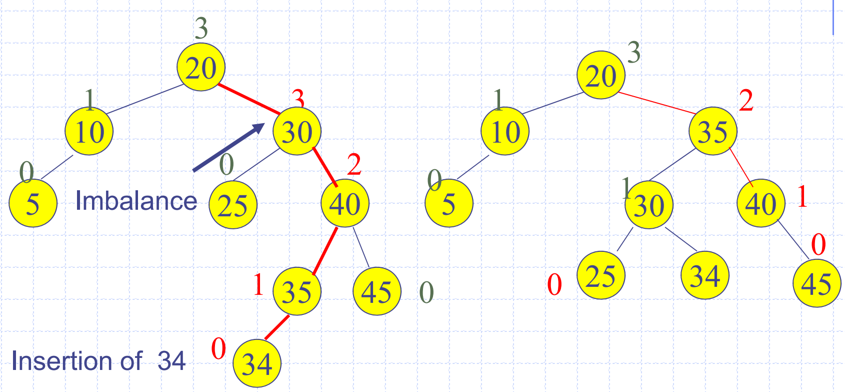
Double rotation (inside case)



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In Class Exercises

- Build an AVL tree with the following values:
  15, 20, 24, 10, 13, 7, 30, 36, 25

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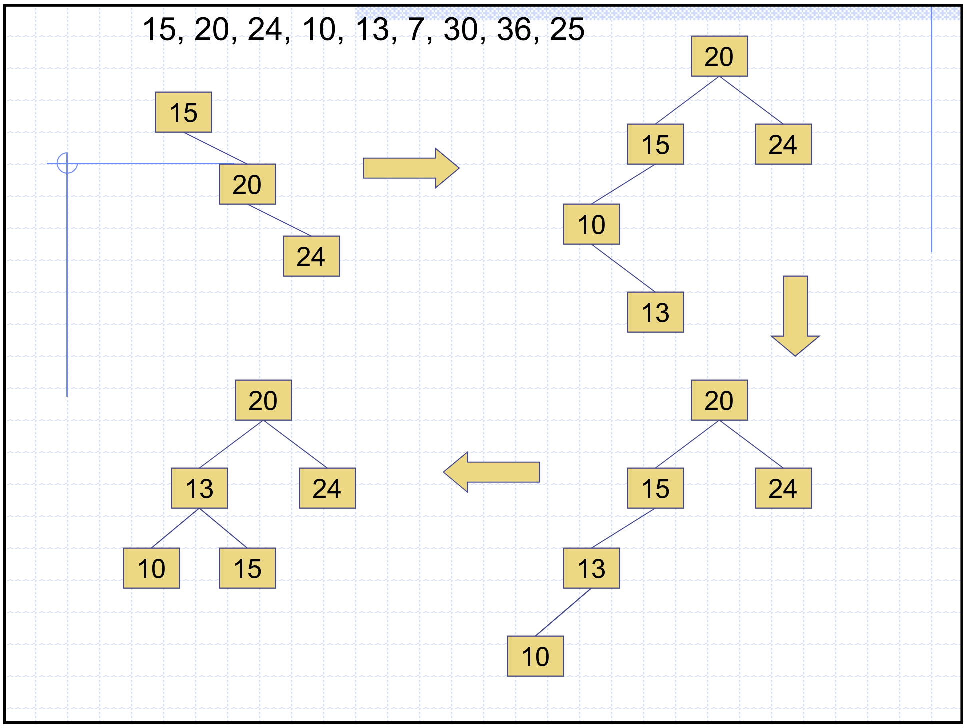

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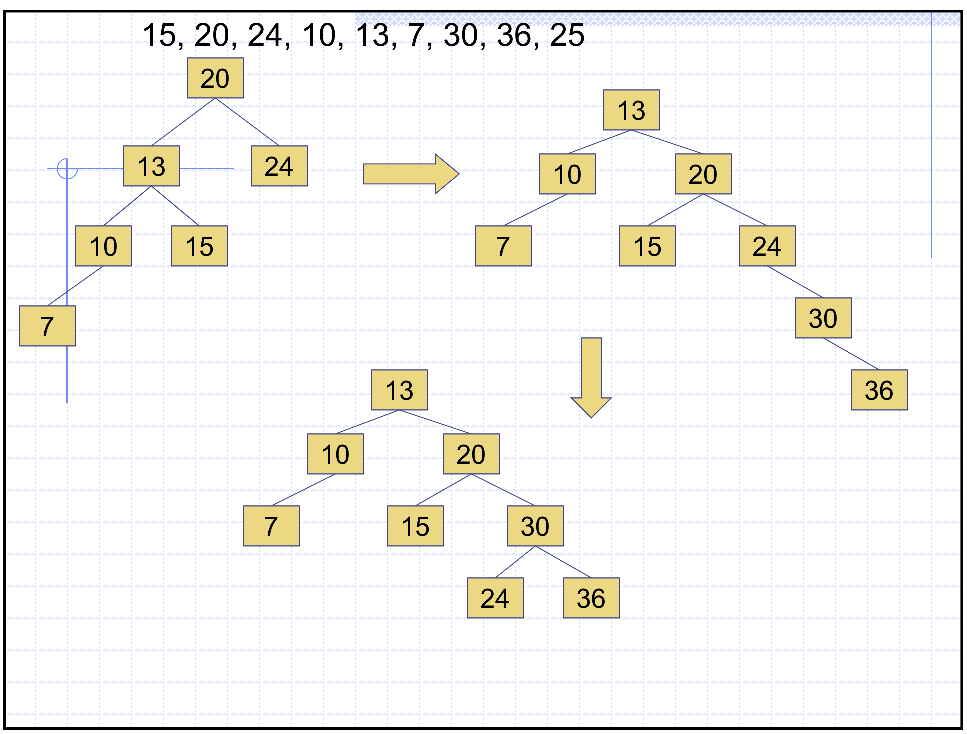

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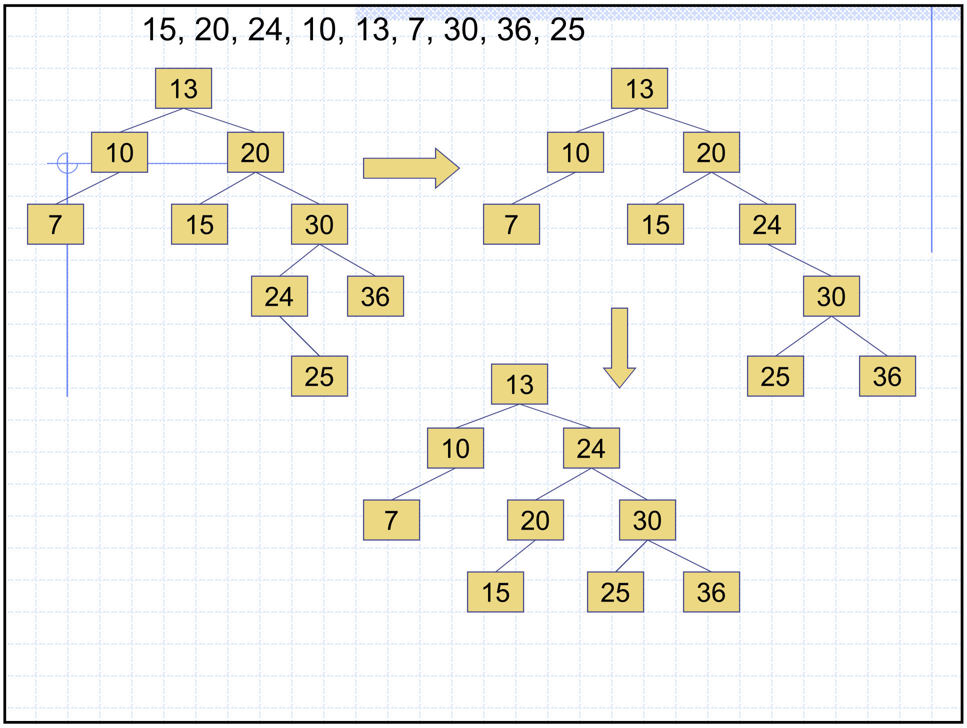

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Possible Quiz Questions

- Build an AVL tree by inserting the following values in the given order:
  1, 2, 3, 4, 5, 6.

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AVL Tree Deletion

- Similar but more complex than insertion
  - Rotations and double rotations needed to rebalance
  - Imbalance may propagate upward so that many rotations may be needed.

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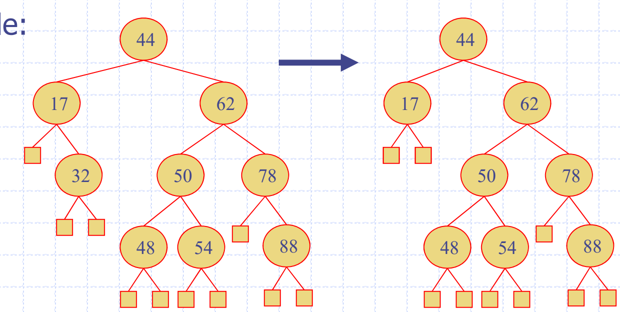
Deletion

- Removal begins as in a binary search tree, which means the node removed will become an empty external node. Its parent may have an imbalance.
- Example:



before deletion of 32

after deletion

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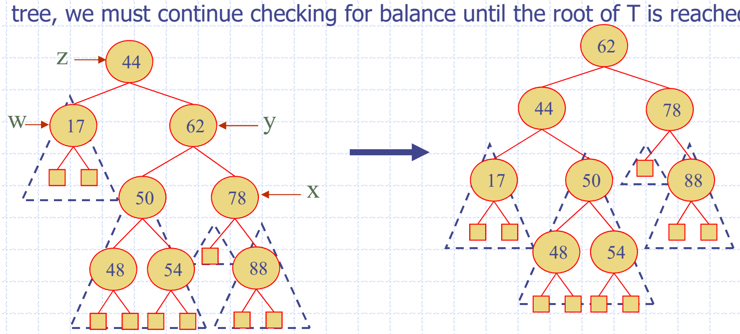
Rebalancing after a Removal

- Let z be the first unbalanced node encountered while travelling up the tree from w. Also, let y be the child of z with the larger height, and let x be the child of y with the larger height
- We perform a rotateToLeft to restore balance at z
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached



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Deletion in standard BST

Algorithm ***remove***(***k***, ***v***)
input: delete the node containing key ***k***
output: the tree without ***k***.
if *isNull* (*v*)
  return *v*
if ***k*** < ***key***(***v***) // duplicate keys are okay
  leftChild(***v***) ← ***remove***(***k***, ***leftChild***(***v***))
else if ***k*** > ***key***(***v***)
  rightChild(***v***) ← ***remove***(***k***, ***rightChild***(***v***))
else if *isNull*(***leftChild***(v))
  return ***rightChild***(***v***)
else if *isNull*(***rightChild***(***v***))
  return ***leftChild***(***v***)
node max ← ***treeMaximum***(***leftChild***(***v***))
key(***v***) ← ***key***(***min***)
rightChild(***v***) ← ***remove***(***key***(***min***), ***rightChild***(***v***))
return ***v***

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Deletion in AVL Trees

```
Algorithm remove(k, v)
  input: delete the node containing key k
  output: the tree without k.
  if isNull (v)
     return v
  if k < key(v)    // duplicate keys are okay
     leftChild(v) ← remove(k, leftChild(v))
  else if k > key(v)
     rightChild(v) ← remove(k, rightChild(v))
  else if isNull(leftChild(v))
     return rightChild(v)
  else if isNull(rightChild(v))
     return leftChild(v)
  node max ← treeMaximum(leftChild(v))
  key(v) ← key(max)
  leftChild(v) ← remove(key(max), leftChild(v))
  AVLbalance(v)
  return v
```

AVLbalance(v)

Assume the height is updated in rotations.

```
if (v.left.height >
    v.right.height+1) {
  y = v.left
  if (y.right.height >
      y.left.height)
     DoubleRotateToRight(v)
  else  rotateToRight(v)
}

if (v.right.height >
    v.left.height+1) {
  y = v.right
  if (y.left.height >
     y.right.height)
     DoubleRotateToLeft(v)
  else  rotateToLeft(v)
}
```

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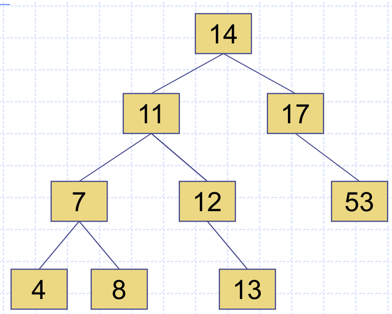
AVL Tree Example:

- **Now remove 53**



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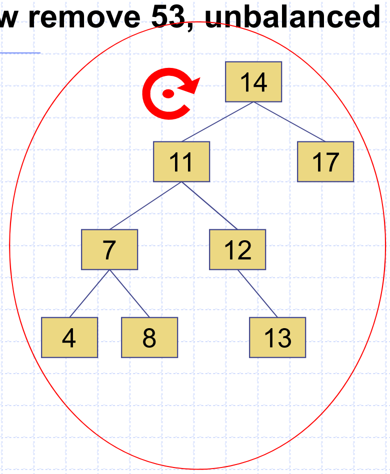
AVL Tree Example:

- **Now remove 53, unbalanced**



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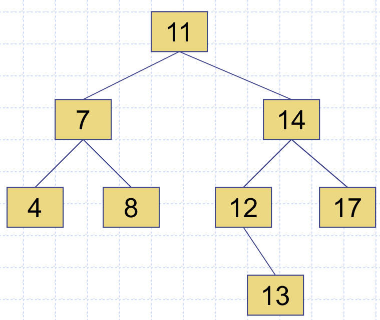
AVL Tree Example:

- **Balanced!**



Now try Remove 11

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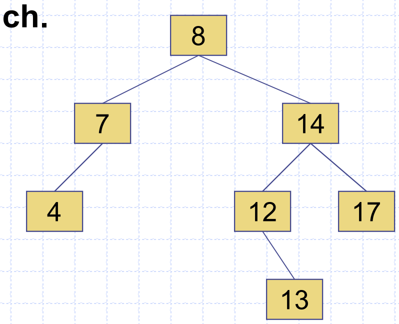
AVL Tree Example:

• Remove 11, replace it with the largest, i.e., 8, in its left branch.



Now try Remove 8.

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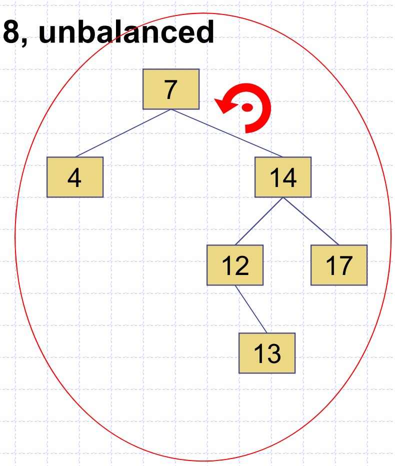
AVL Tree Example:

• Remove 8, unbalanced



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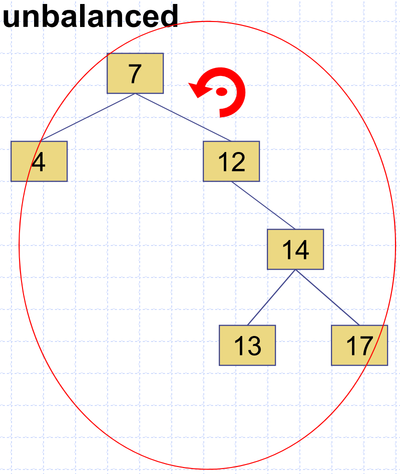
AVL Tree Example:

• Remove 8, unbalanced



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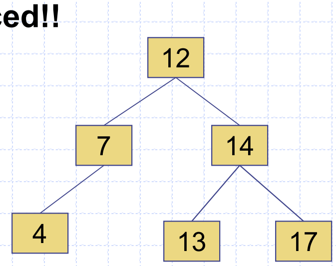
AVL Tree Example:

• Balanced!!



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Deletion in AVL Trees

```
Algorithm remove(k, v)
  input: delete the node containing key k
  output: the tree without k.
  if isNull (v)
     return v
  if k < key(v)    // duplicate keys are okay
     leftChild(v) ← remove(k, leftChild(v))
  else if k > key(v)
     rightChild(v) ← remove(k, rightChild(v))
  else if isNull(leftChild(v))
     return rightChild(v)
  else if isNull(rightChild(v))
     return leftChild(v)
  node max ← treeMaximum(leftChild(v))
  key(v) ← key(max)
  leftChild(v) ← remove(key(max), leftChild(v))
  return AVLbalance(v)
```

```
AVLbalance(v) {
Assume the height is updated in rotations.

if (v.left.height >
   v.right.height+1) {
  y = v.left
  if (y.right.height >
      y.left.height)
     v = DoubleRotateToRight(v)
  else  v = rotateToRight(v)
}

if (v.right.height >
   v.left.height+1) {
  y = v.right
  if (y.left.height >
     y.right.height)
     v = DoubleRotateToLeft(v)
  else  v = rotateToLeft(v)
}
return v
}
```

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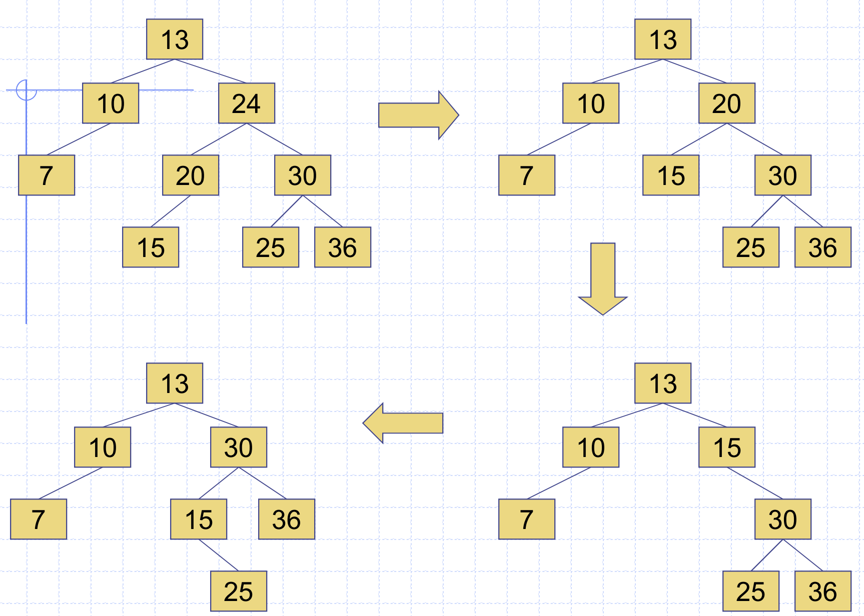
Remove 24 and 20 from the AVL tree.



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AVL Tree Performance

- AVL tree storing n items
  - The data structure uses O(n) space
  - A single restructuring takes O(1) time
    - using a linked-structure binary tree
  - Searching takes O(log n) time
    - height of tree is O(log n), no restructures needed
  - Insertion takes O(log n) time
    - initial find is O(log n)
    - restructuring up the tree, maintaining heights is O(log n)
  - Removal takes O(log n) time
    - initial find is O(log n)
    - restructuring up the tree, maintaining heights is O(log n)

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Pros and Cons of AVL Trees

Arguments for AVL trees:

1. Search is O(log N) since AVL trees are always balanced.
2. Insertion and deletions are also O(logn)
3. The height balancing adds no more than a constant factor to the speed of insertion.

Arguments against using AVL trees:

1. Difficult to program & debug; more space for height.
2. Asymptotically faster but rebalancing costs time.
3. Most large searches are done in database systems on disk and use other structures (e.g. B-trees).
4. May be OK to have O(N) for a single operation if the total run time for many consecutive operations is fast (e.g. Splay trees).

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Red-Black Tree

- A ref-black tree is a binary search such that each node has a color of either red or black.
- The root is black.
- Empty (or null) nodes are assumed black.
- Every path from a node to a leaf contains the same number of black nodes.
- If a node is red then its parent must be black.

```
Class BinaryNode
    KeyType: Key
    Boolean: isRed
    BinaryNode: LeftChild
    BinaryNode: RightChild

   Constructor(KeyType: key)
        Key = key
        isRed = true
   End Constructor
End Class
```

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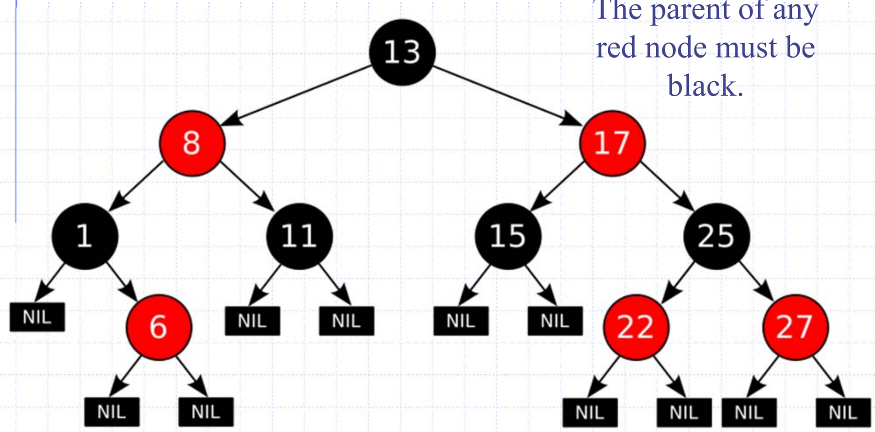
Example

The root is black.

The parent of any red node must be black.



62

Theorem: Any red-black tree with root ***x***, has **$n \geq 2^{h/2} - 1$** nodes, where h is the height of tree rooted by x.

Proof: We repeatedly replace the subtree rooted by a red node by one of its children.

Let the height of the new tree be h', then h' >= h/2, because the number of red nodes in any path is no more than the number of black nodes.

The new tree is a perfect binary tree, because it has the same of nodes from the root to any leaf. It must have $2^{h'} - 1$ nodes.

So $h \leq 2\log(n+1)$.

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Maintain the Red Black Properties in a Tree

- Insertions
  - Must maintain rules of Red Black Tree.
  - New Node always added at leaf
  - can't be black or we will violate rule of the same # of blacks along any path
  - therefore the new node must be red
  - If parent is black, done (trivial case)
  - If parent red, things get interesting because a red node with a red parent violates no double red rule.

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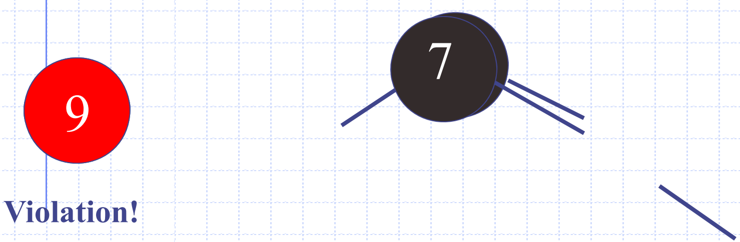
Algorithm: Insertion

A red-black tree is a particular binary search tree, so create a new node as red and insert it as in normal search tree.



Violation!

What property is violated?

The parent of a red node must be black.

Solution: (1) Rotate; (2) Switch colors.

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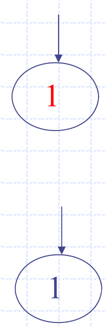
Example of Inserting Sorted Numbers

- 1 2 3 4 5 6 7 8 9 10

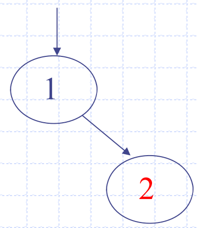


Insert 1. A leaf is red. Realize it is root so recolor to black.

66

Insert 2

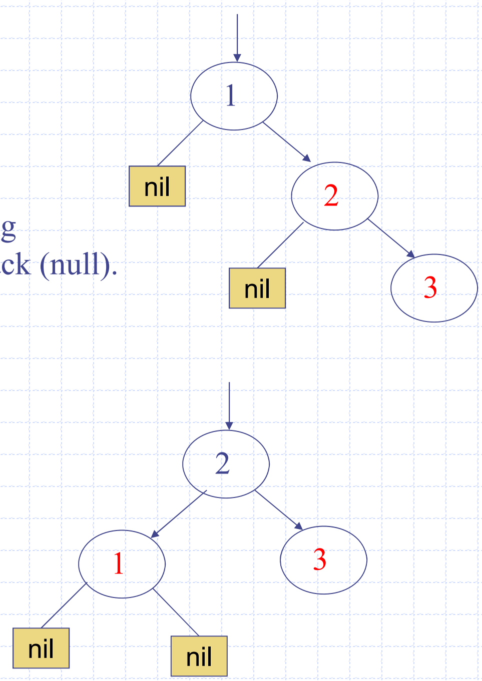
make 2 red. Parent is black so done.



67

Insert 3

Insert 3. Parent is red. 2's uncle, i.e., the sibling of the parent of 2, is black (null). 3 is outside relative to grandparent. Rotate parent and grandparent



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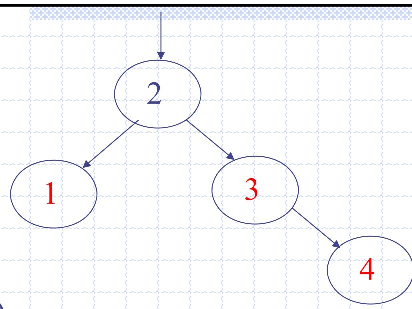
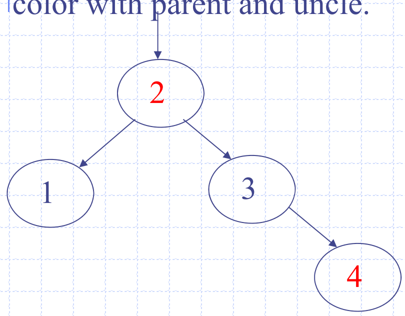
Insert 4

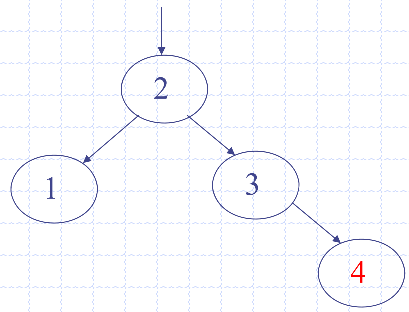


When adding 4
parent is red.

4 has a red uncle (1).
So switch the great parent (2)'s
color with parent and uncle.

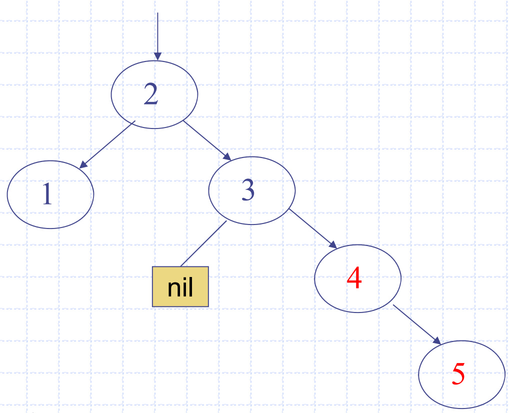


2 is set to black if it's the root.



69

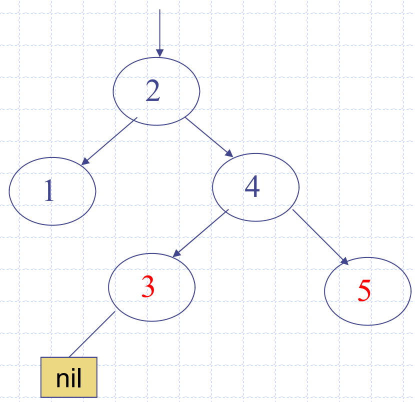
Insert 5



5's parent is red.
5's uncle is
black (null).
5 is outside relative to
grandparent (3) so rotate
parent and grandparent then
recolor

70

Finish insert of 5



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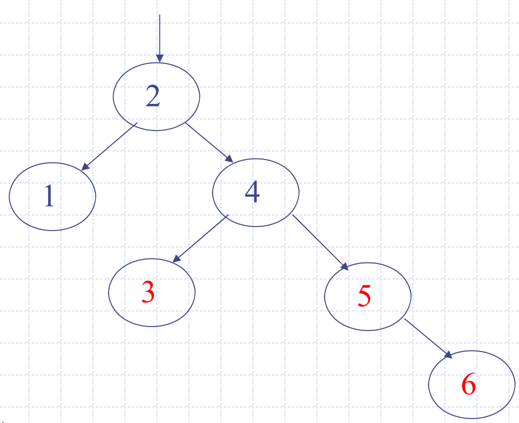
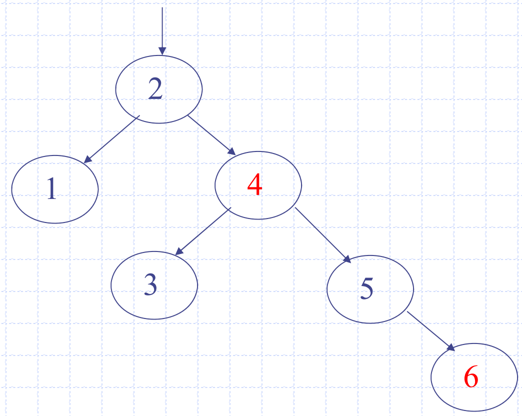
Insert 6



6 has a red uncle (3).
So switch the grandparent (4)'s
color with parent (5) and uncle (3).

72

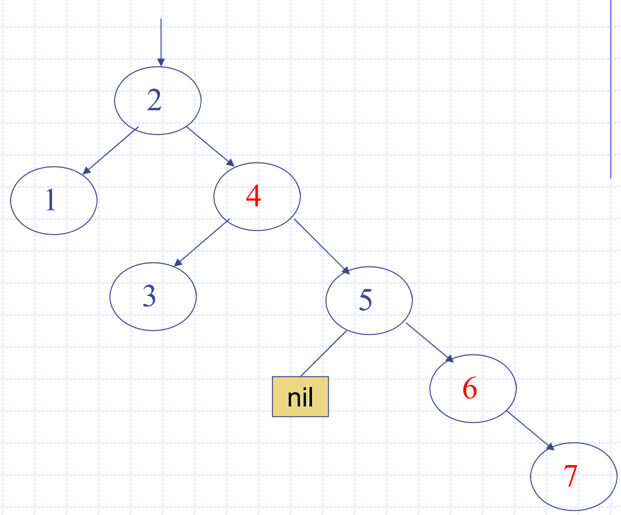
Finishing insert of 6



4's parent is black so done.

73

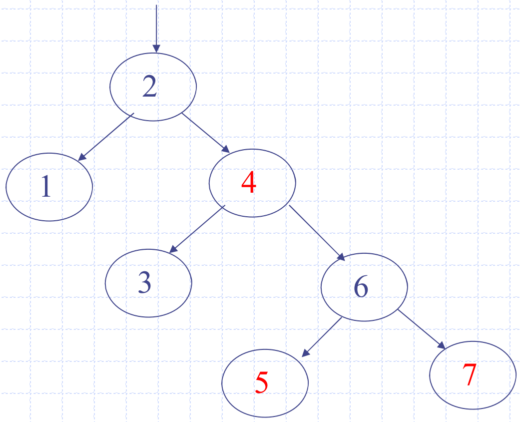
Insert 7



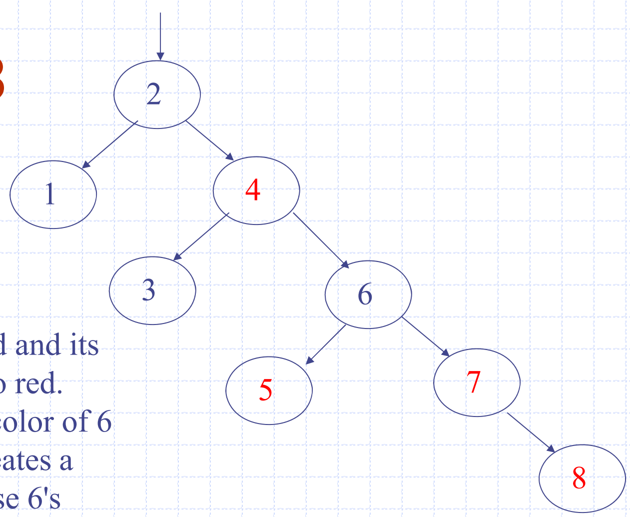
7's parent is red. Parent's sibling is black (null). 7 is outside relative to grandparent (5) so rotate parent and grandparent then recolor

74

Finish insert of 7

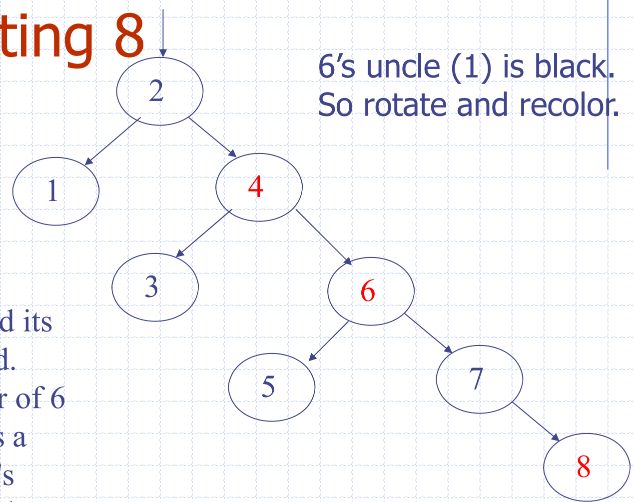


Insert 8



8’s parent is red and its uncle (5) is also red. Switching the color of 6 with 5 and 7creates a problem because 6's parent, 4, is also red. Must handle the red-red violation at 6.

Still Inserting 8



6’s uncle (1) is black. So rotate and recolor.

8’s parent is red and its uncle (5) is also red. Switching the color of 6 with 5 and 7creates a problem because 6's parent, 4, is also red. Must handle the red-red violation at 6.

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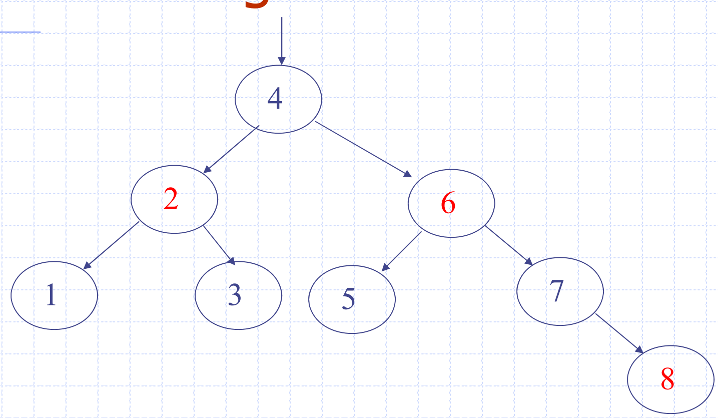
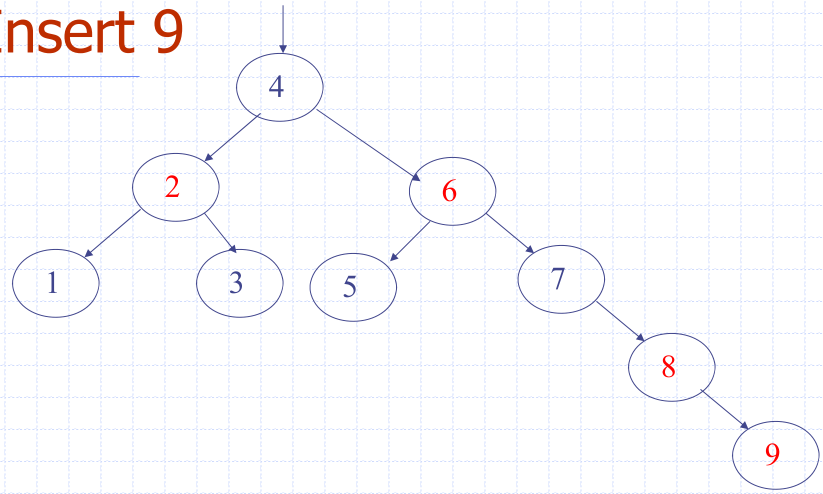
Finish inserting 8



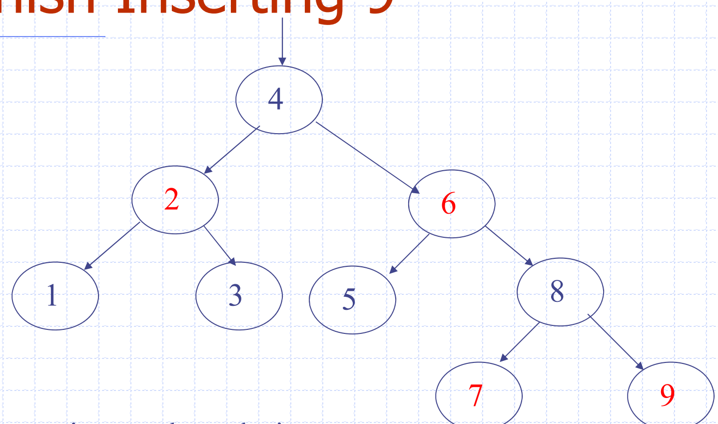
78

Insert 9



79

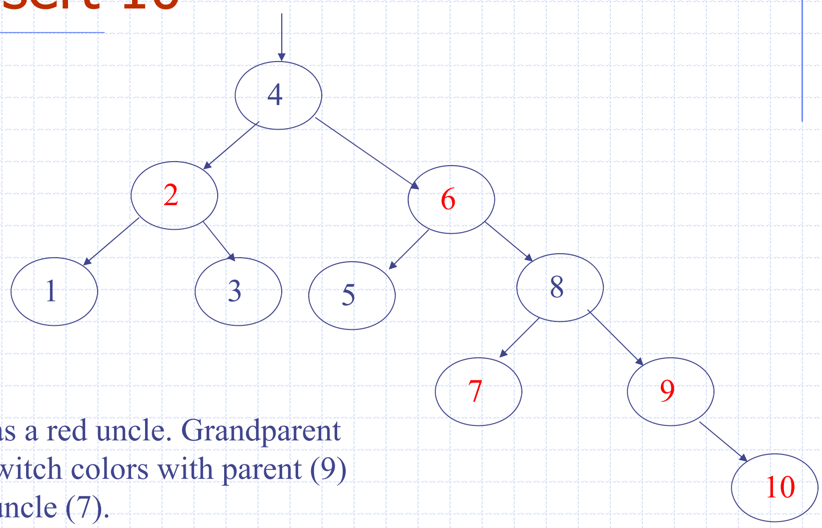
Finish Inserting 9



After rotations and recoloring

80

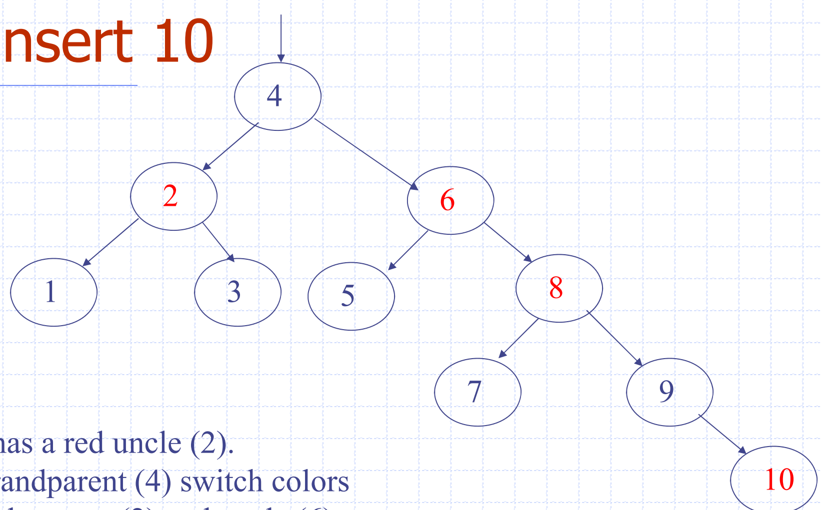
Insert 10



10 has a red uncle. Grandparent (8) switch colors with parent (9) and uncle (7).

81

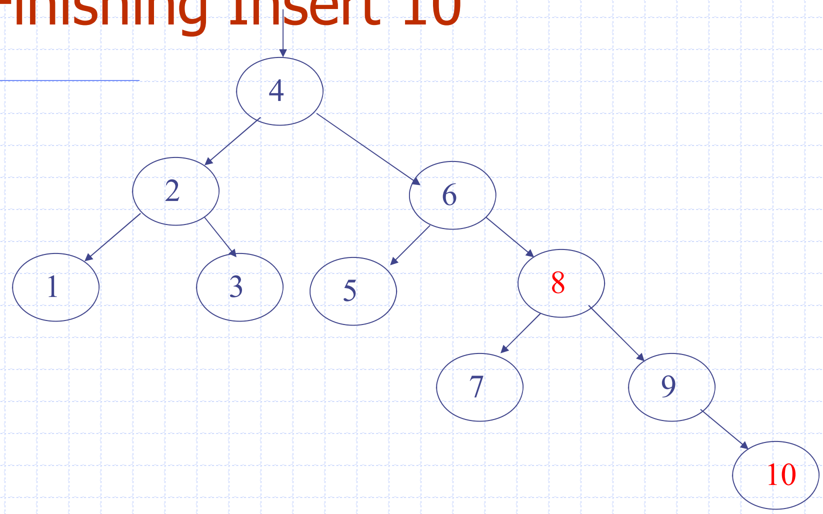
Insert 10



8 has a red uncle (2). Grandparent (4) switch colors with parent (2) and uncle (6). 4 is recolored black as root.

82

Finishing Insert 10

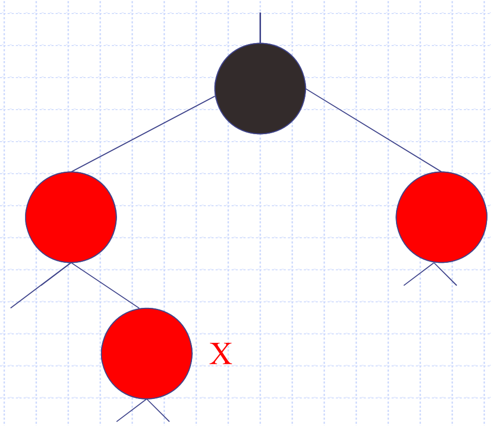


83

Algorithm: Insertion

We have detected a need for balance when *X* is red and its parent, too.

• If X has a red uncle: colour the parent and uncle black, and grandparent red. Then replace X by grandparent to see if new X's parent is red.

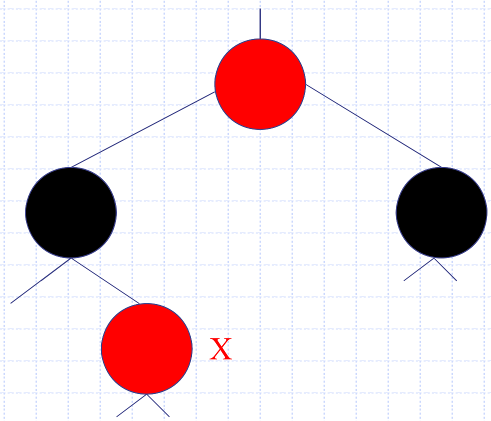


84

Algorithm: Insertion

We have detected a need for balance when *X* is red and its parent, too.

• If X has a red uncle: colour the parent and uncle black, and grandparent red. Then replace X by grandparent to see if new X's parent is red.

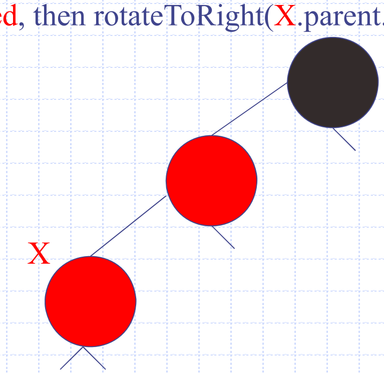


Algorithm: Insertion

We have detected a need for balance when X is red and his parent too.

• If X has a red uncle: colour the parent and uncle black, and grandparent red. Then replace X by grandparent to see if new X's parent is red.

• If X is a left child and has a black uncle: colour the parent black and the grandparent red, then rotateToRight(X.parent.parent)



Algorithm: Insertion

We have detected a need for balance when X is red and his parent too.

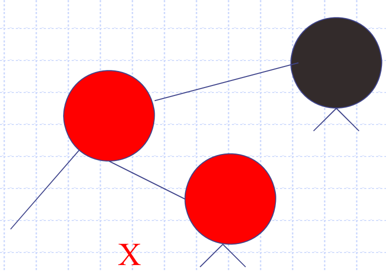
• If X has a red uncle: colour the parent and uncle black, and grandparent red. Then replace X by grandparent to see if X's parent is red.
• If X is a left child and has a black uncle: colour the parent black and the grandparent red, then rotateRight(X.parent.parent)

87

Algorithm: Insertion

We have detected a need for balance when X is red and his parent too.

• If X has a red uncle: colour the parent and uncle black, and grandparent red. Then replace X by grandparent to see if X's parent is red.
• If X is a left child and has a black uncle: colour the parent black and the grandparent red, then rotateRight(X.parent.parent)
• If X is a right child and has a black uncle, then rotateLeft(X.parent) and

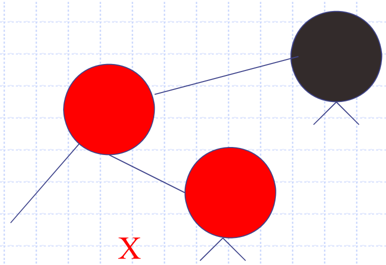


88

Algorithm: Insertion

We have detected a need for balance when X is red and his parent too.

- If X has a red uncle: colour the parent and uncle black, and grandparent red. Then replace X by grandparent to see if X's parent is red.
- If X is a right child and has a black uncle, then rotateToLeft(X.parent) and
- If X is a left child and has a black uncle: colour the parent black and the grandparent red, then rotateToRight(X.parent.parent)



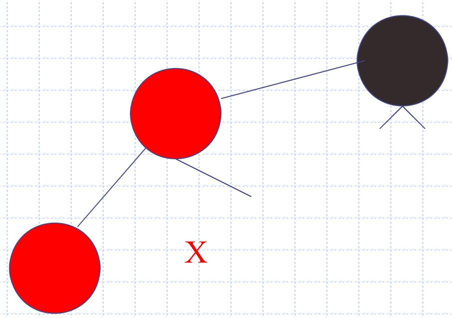
89

Algorithm: Insertion

We have detected a need for balance when X is red and his parent too.

- If X has a red uncle: colour the parent and uncle black, and grandparent red. Then replace X by grandparent to see if X's parent is red.

- If X is a right child and has a black uncle, then rotateLeft(X.parent) and
- If X is a left child and has a black uncle: colour the parent black and the grandparent red, then rotateRight(X.parent.parent)



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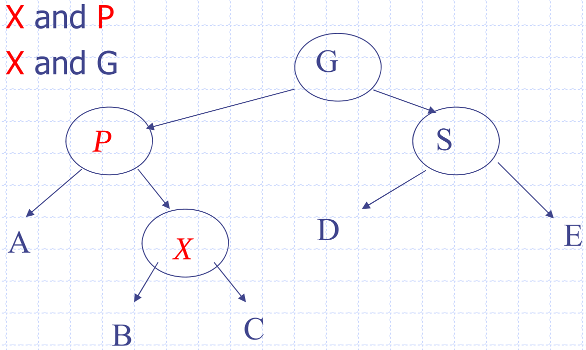
Double Rotation

- What if X is at left right relative to G?
  - a single rotation will not work
- Must perform a double rotation
  - rotate X and P
  - rotate X and G



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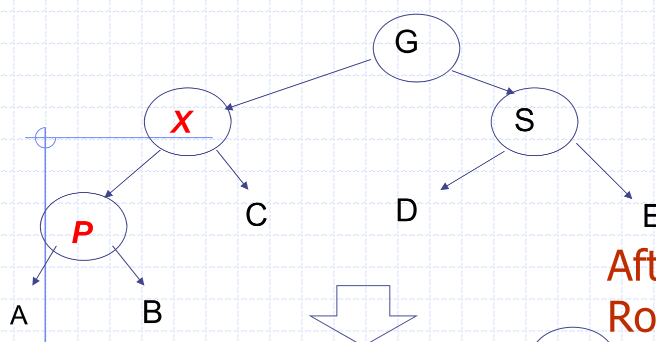
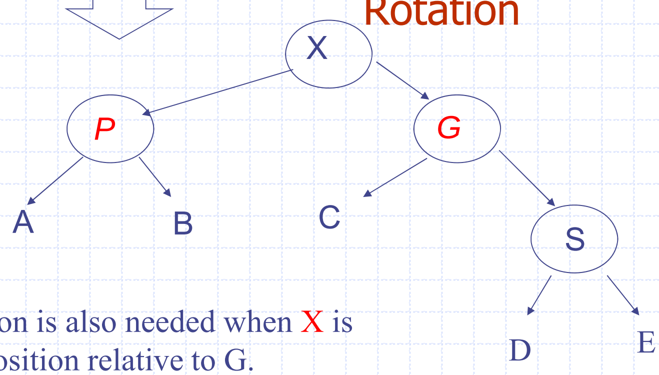


After Double Rotation



Double rotation is also needed when X is at right left position relative to G.

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Properties of Red Black Trees

- If a Red node has any children, it must have two children and they must be black. (Why?)
- If a black node has only one child, that child must be a Red leaf. (Why?)
- Due to the rules there are limits on how unbalanced a Red Black tree may become.

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Red Black Trees vs AVL Trees

- AVL trees provide **faster lookups** than Red Black Trees because they are more strictly balanced.
- Red Black Trees provide **faster insertion and removal** operations than AVL trees as fewer rotations are done due to relatively relaxed balancing.
- AVL trees store **balance factors or heights** with each node, thus requires storage for an integer per node whereas Red Black Tree requires only 1 bit of information per node.

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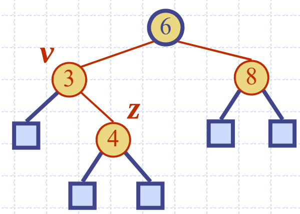
Splay Trees



95

95

Motivation for Splay Trees

Problems with AVL Trees
- extra storage/complexity for height fields
- ugly delete code

Solution: splay trees
- blind adjusting version of AVL trees
- amortized time for all operations is O(log n)
- worst case time is O(n)
- insert/find always rotates node *to the root*!

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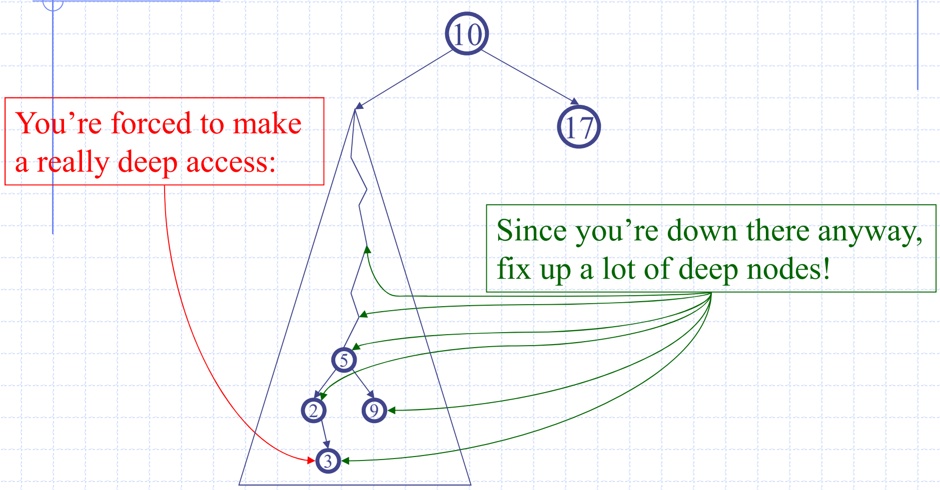
Splay Tree Idea



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Splaying Cases

Node n being accessed is:

- Root
- Child of root
- Has both parent (p) and grandparent (g)
  - Zig-zig pattern: g → p → n is left-left or right-right (outside nodes)
  - Zig-zag pattern: g → p → n is left-right or right-left (inside nodes)

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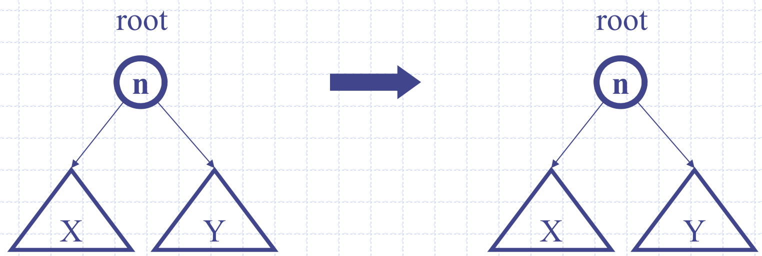
Access root:

Do nothing (that was easy!)



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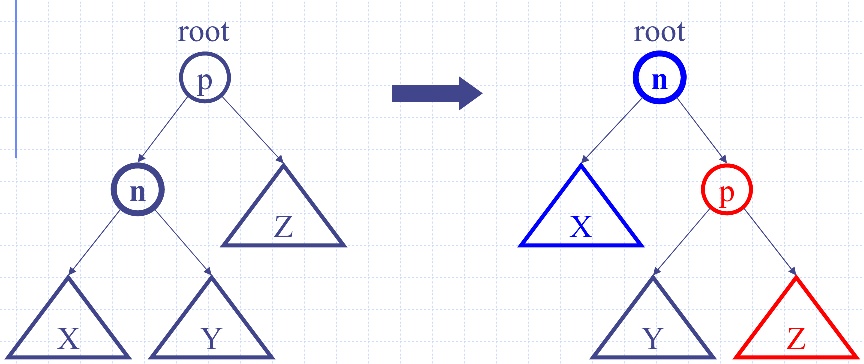
Access child of root:

Zig (AVL single rotation)



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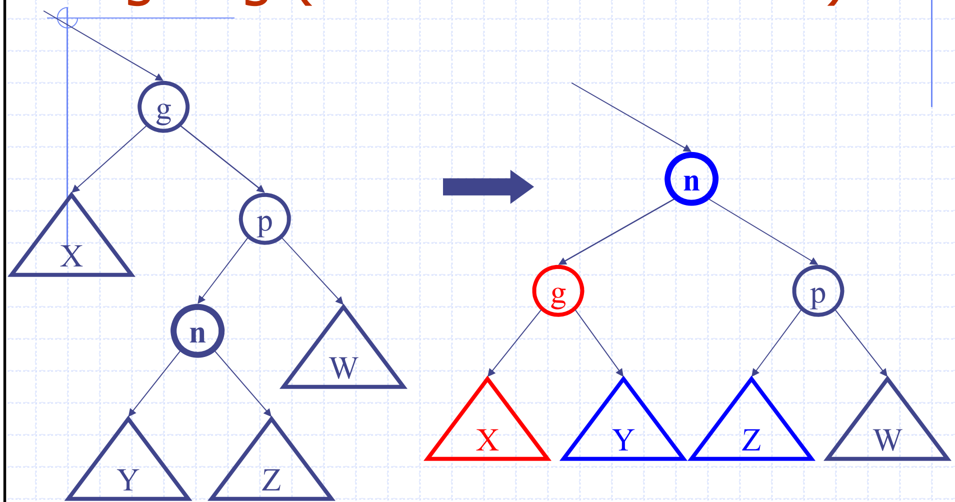
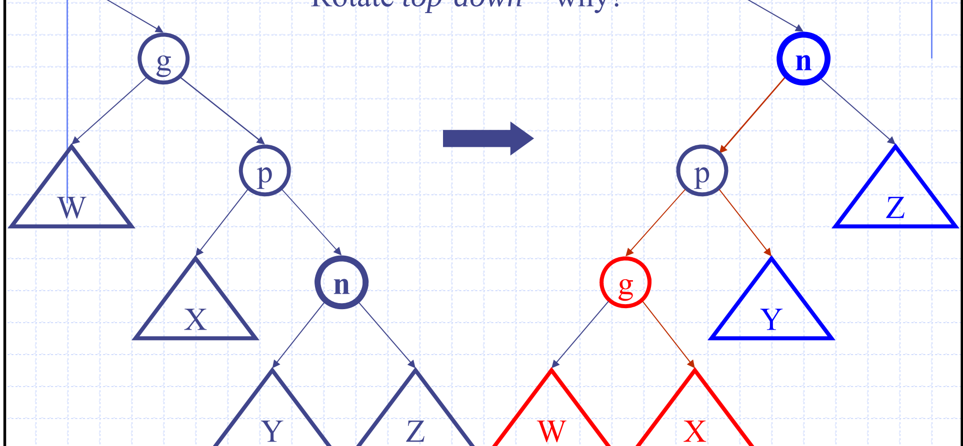
Access (LR, RL) grandchild:
Zig-Zag (AVL double rotation)



101

Access (LL, RR) grandchild:
Zig-Zig

Rotate *top-down* – why?



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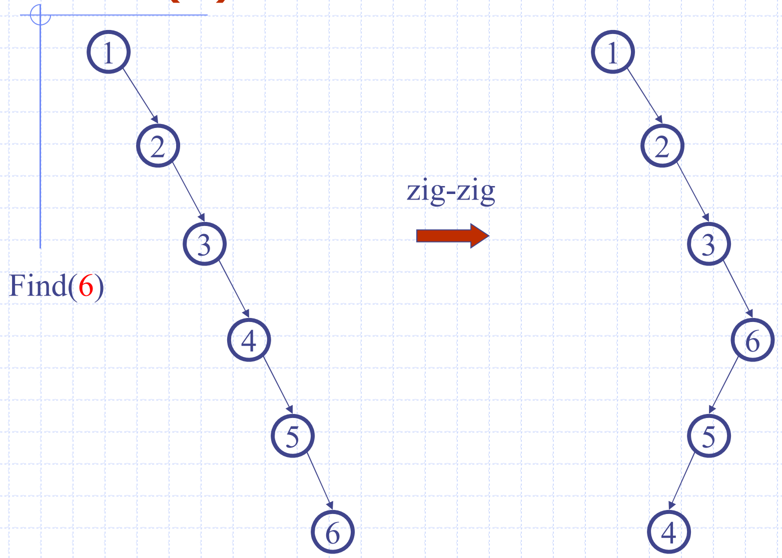
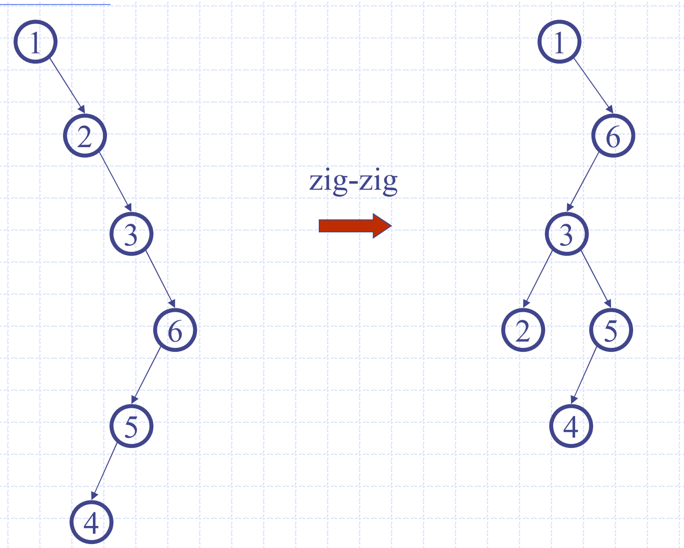
Splaying Example: Find(6)



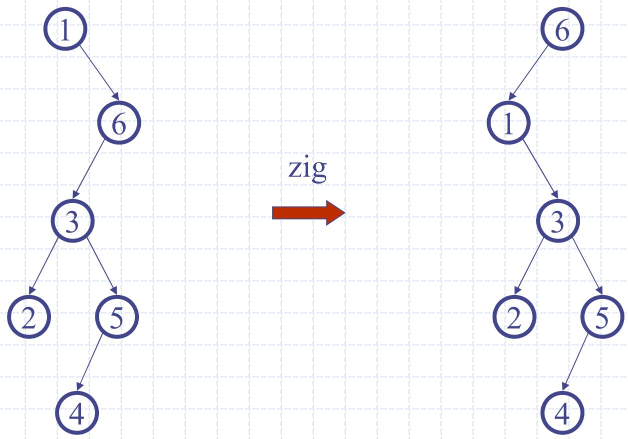
103

… still splaying …



104

… 6 splayed out!



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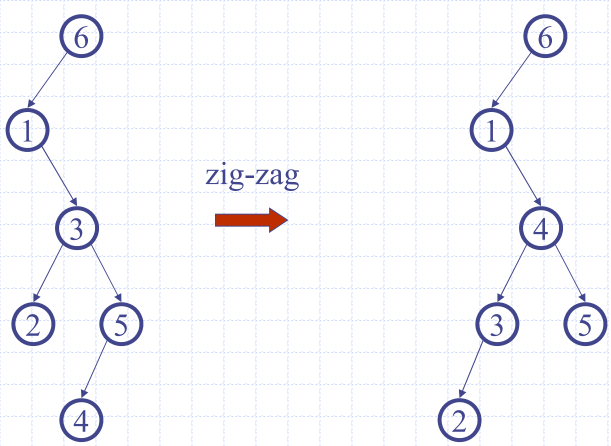
Splay it Again!
Find (4)

Find(4)



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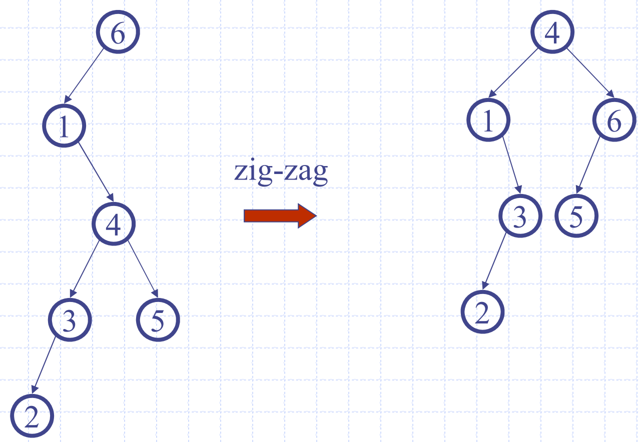
... 4 splayed out!



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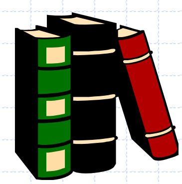
Splay Tree Definition



- A **splay tree** is a binary search tree where a node is splayed after it is accessed (for a search or update)
  - deepest internal node accessed is splayed
  - splaying costs O(h), where h is height of the tree – which is still O(n) worst-case
    - O(h) rotations, each of which is O(1)

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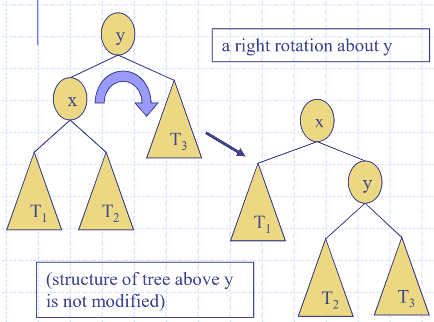
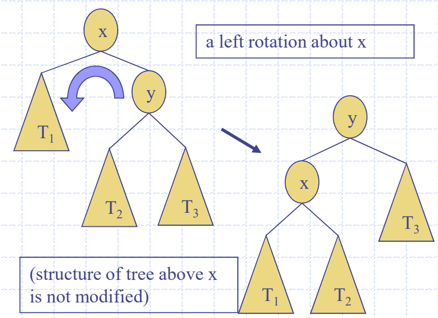
Splay Trees do Rotations after Every Operation (Even Search)

- new operation: ***splay***
  - splaying moves a node to the root using rotations

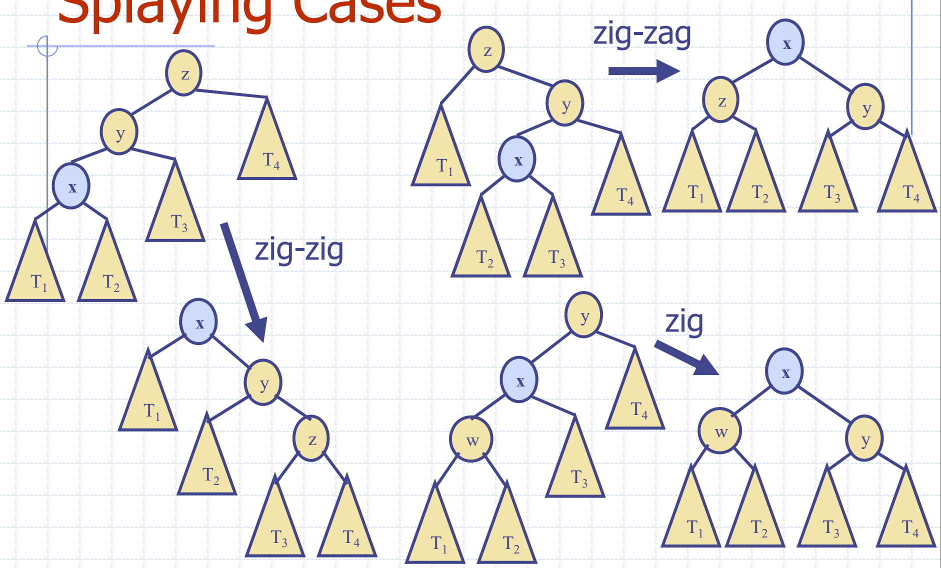
- right rotation
  - makes the left child x of a node y into y' s parent; y becomes the right child of x



- left rotation
  - makes the right child y of a node x into x' s parent; x becomes the left child of y

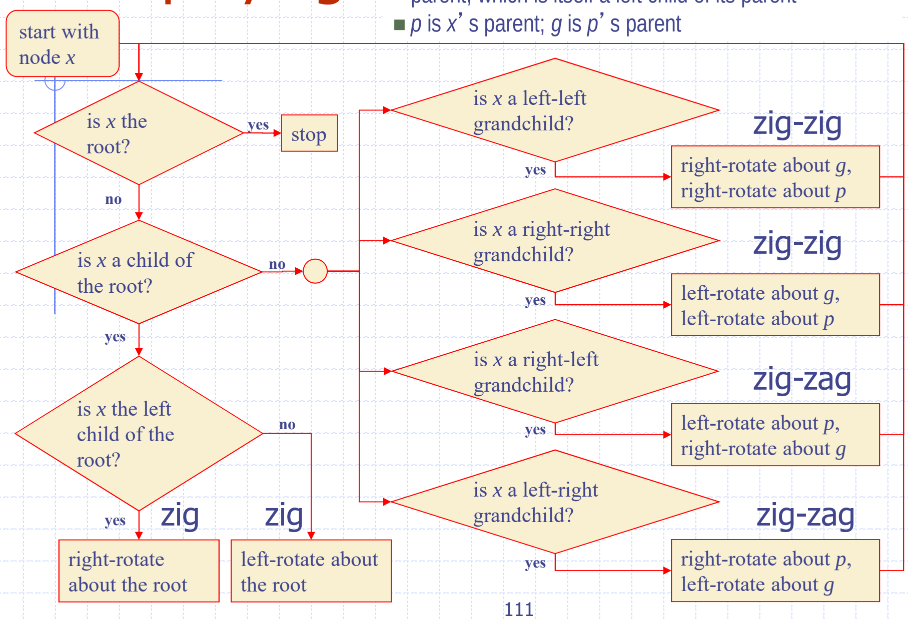


Visualizing the Splaying Cases



Splaying:

- "*x* is a left-left grandchild" means *x* is a left child of its parent, which is itself a left child of its parent
- *p* is *x*'s parent; *g* is *p*'s parent



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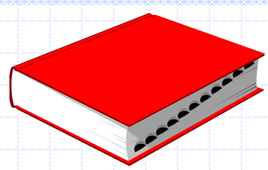
Splay Tree Operations



- Which nodes are splayed after each operation?

method	splay node
Search for k	if key found, use that node if key not found, use parent of ending external node
Insert (k,v)	use the new node containing the entry inserted
Remove item with key k	use the predecessor of the node to be removed

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Why Splaying Helps

- If a node *n* on the access path is at depth *d* before the splay, it's at about depth *d/2* after the splay
  - Exceptions are the root, the child of the root, and the node splayed
- Overall, nodes which are below nodes on the access path tend to move closer to the root
- Splaying gets amortized O(log n) performance. (Maybe not now, but soon, and for the rest of the operations.)

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Splay Operations: Find

- Find the node in normal BST manner
- Splay the node to the root

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Splay Operations: Insert

- Ideas?
- Can we just do BST insert?

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Digression: Splitting

- Split(T, x) creates two BSTs L and R:
  - all elements of T are in either L or R ($T = L \cup R$)
  - all elements in L are $\leq x$
  - all elements in R are $\geq x$
  - L and R share no elements ($L \cap R = \varnothing$)

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Splitting in Splay Trees

How can we split?

- We have the splay operation.
- We can find x or the parent of where x should be.
- We can splay it to the root.
- Now, what's true about the left subtree of the root?
- And the right?

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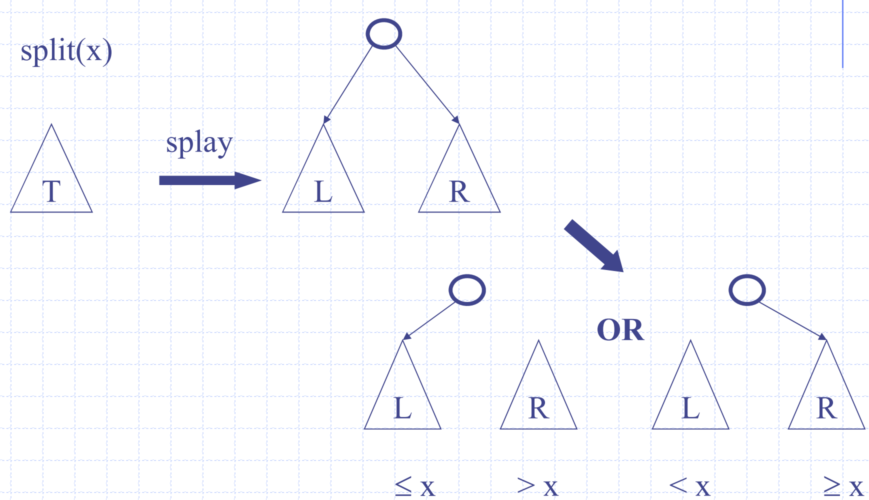
Split



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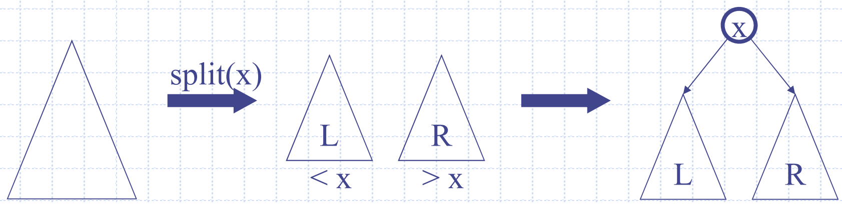
Back to Insert




```
void insert(Node root, Object x)
{
  <left, right> = split(root, x);
  root = newNode(x, left, right);
}
```

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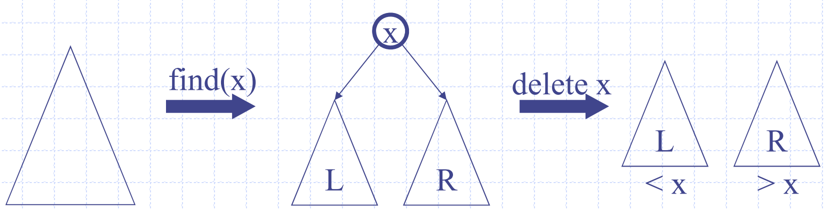
Splay Operations: Delete



Now what?

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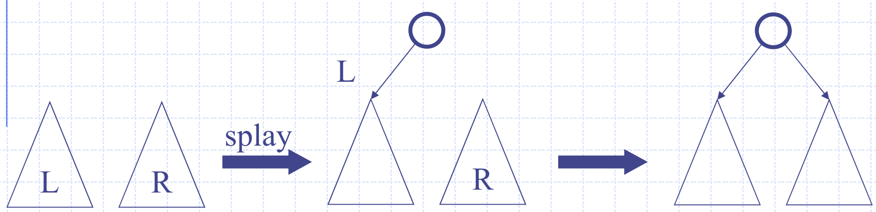
Join

Join(L, R): given two trees such that L < R, merge them



Splay on the maximum element in L, then attach R

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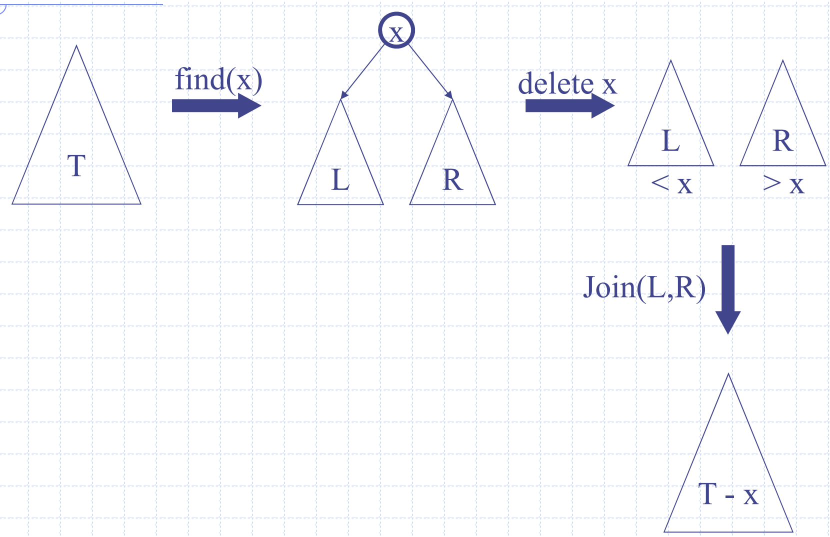
Delete Completed



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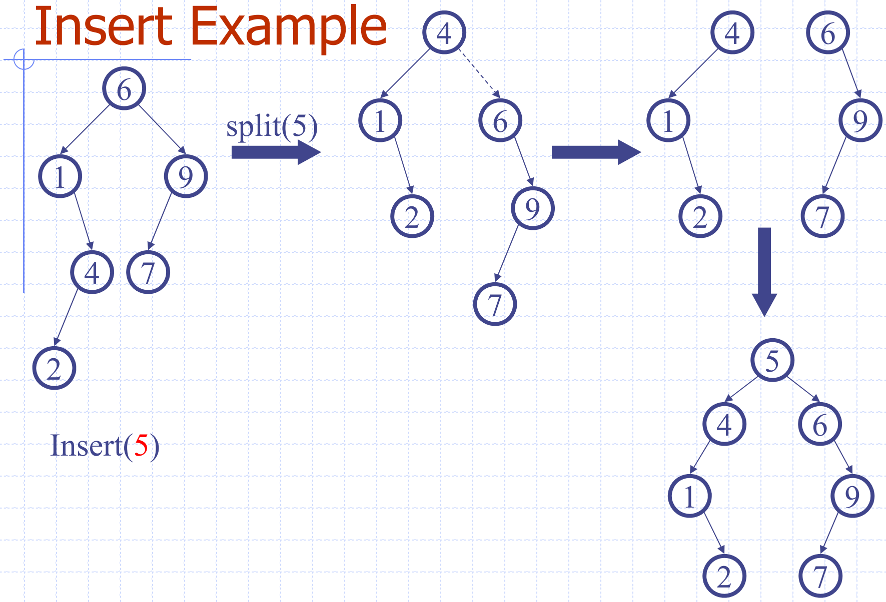
Insert Example



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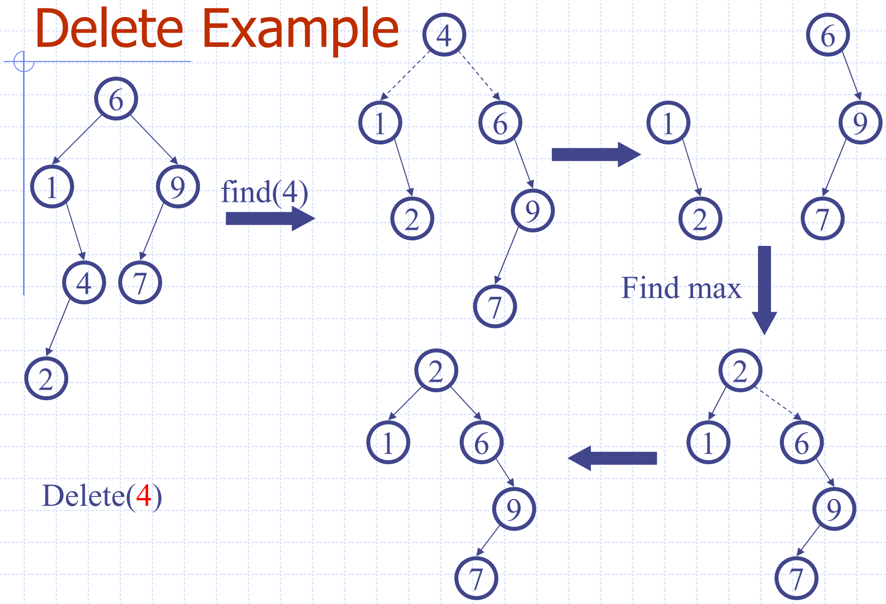
Delete Example



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Splay Tree Summary

Can be shown that any m consecutive operations starting from an empty tree take at most O(m log(n)), where n is the total number of elements in the tree.

→ All splay tree operations run in amortized O(log n) time

O(N) operations can occur, but splaying makes them infrequent

Implements most-recently used (MRU) logic
- Splay tree structure is self-tuning

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Splay Tree Summary (cont.)

Splaying can be done top-down; better because:
- only one pass
- no recursion or parent pointers necessary

There are alternatives to split/insert and join/delete

Splay trees are *very* effective search trees
- relatively simple: no extra fields required
- excellent **locality** properties:
  - frequently accessed keys are cheap to find (near top of tree)
  - infrequently accessed keys stay out of the way (near bottom of tree)

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Amortized Analysis of Splay Trees

- Running time of each operation is proportional to time for splaying.
- Define rank(v) as the logarithm (base 2) of the number of nodes in subtree rooted at v:
  - rank(v) = log n(v) if null for external nodes
  - rank(v) = log (2n(v)+1) if empty nodes for externals.
- Costs: zig = $1, zig-zig = $2, zig-zag = $2.
- Thus, cost for splaying a node at depth d = $d.
- Imagine that we store rank(v) cyber-dollars at each node v of the splay tree (just for the sake of analysis).
- The total counter values is rank(T) = sum of rank(v) for any node v in T.

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