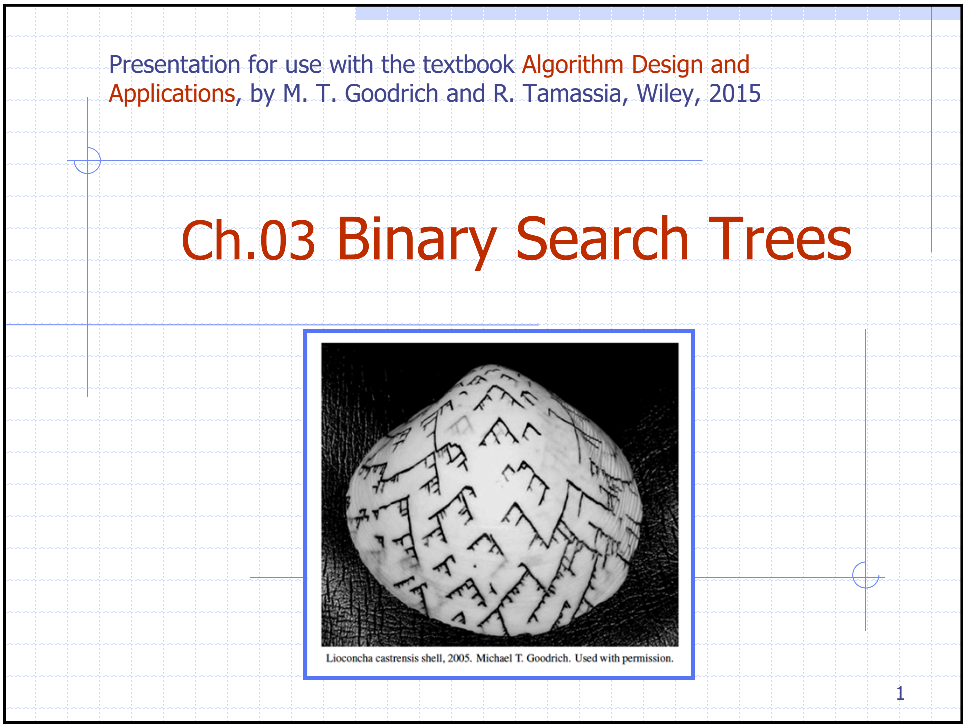
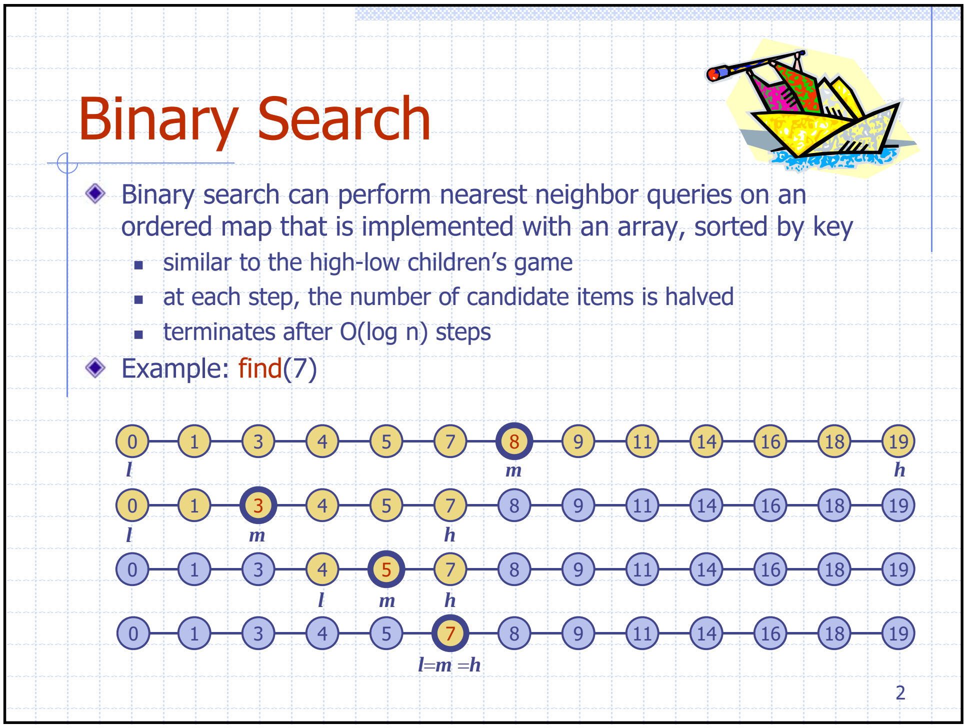


Presentation for use with the textbook Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Ch.03 Binary Search Trees

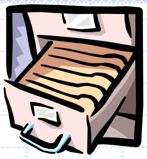
Lioconcha castrensis shell, 2005. Michael T. Goodrich. Used with permission.



Binary Search

- Binary search can perform nearest neighbor queries on an ordered map that is implemented with an array, sorted by key
  - similar to the high-low children’s game
  - at each step, the number of candidate items is halved
  - terminates after O(log n) steps
- Example: find(7)

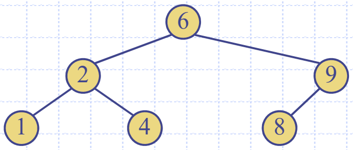
Search Tables

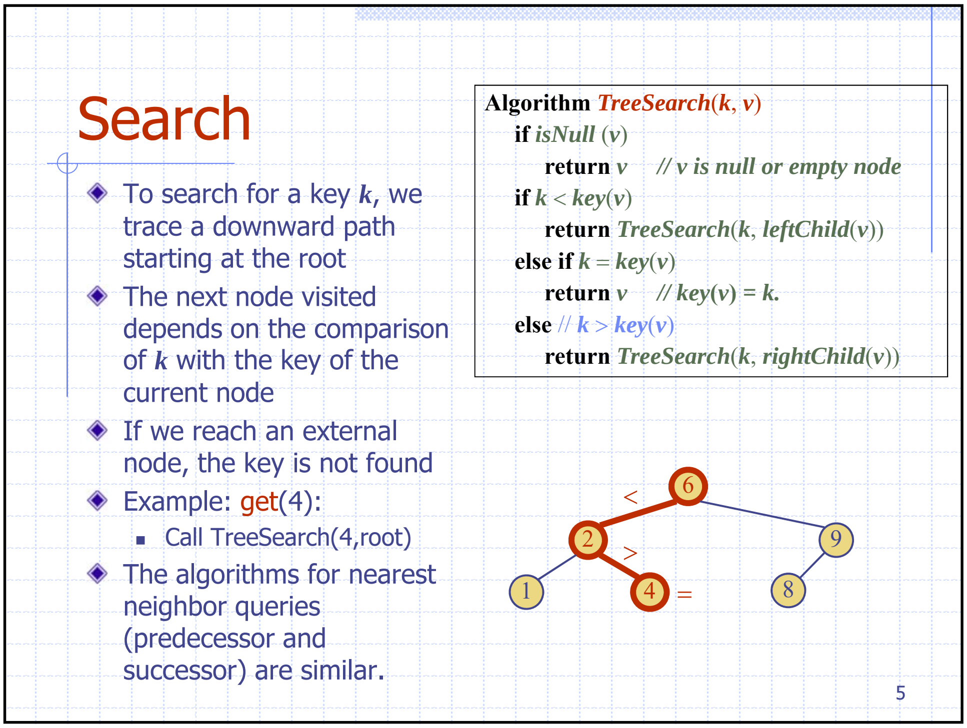


- A search table is an ordered map implemented by means of a sorted sequence
  - We store the items in an array-based sequence, sorted by key
  - We use an external comparator for the keys
- Performance:
  - Searches take $O(\log n)$ time, using binary search
  - Inserting a new item takes $O(n)$ time, since in the worst case we have to shift $n - 1$ items to make room for the new item
  - Removing an item takes $O(n)$ time, since in the worst case we have to shift n items to compact the items after the removal
- The lookup table is effective only for ordered maps of small size or for maps on which searches are the most common operations, while insertions and removals are rarely performed.

Binary Search Trees

- A binary search tree is a binary tree storing keys (or key-value entries) at its internal nodes and satisfying the following property:
  - Let u, v, and w be three nodes such that u is in the left subtree of v and w is in the right subtree of v. We have $key(u) \leq key(v) \leq key(w)$
- An inorder traversal of a binary search trees visits the keys in non-decreasing order

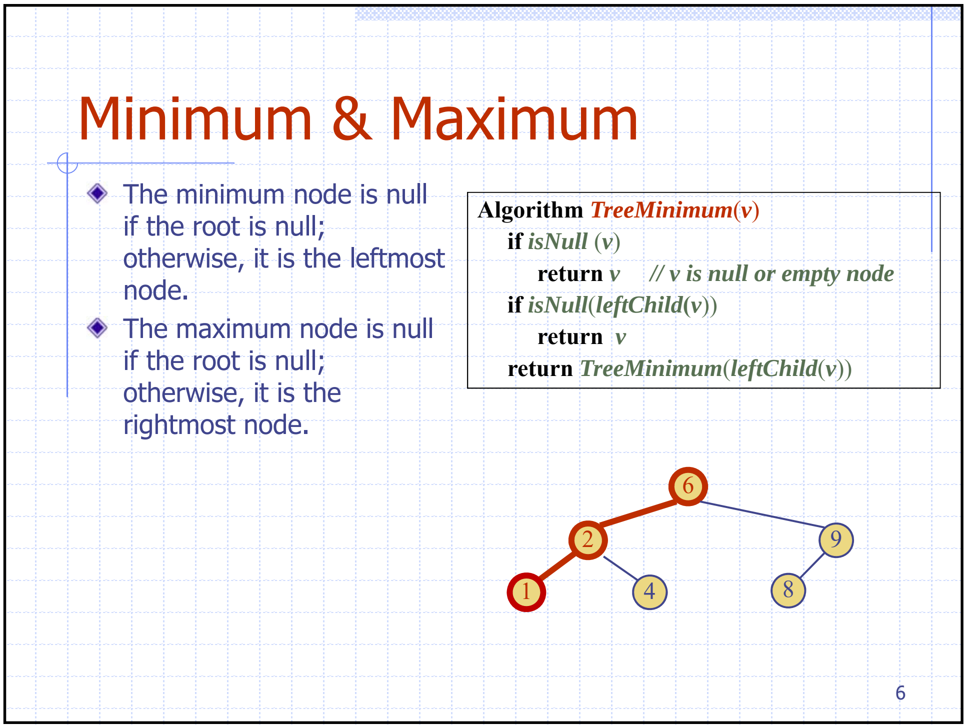




Search

- To search for a key ***k***, we trace a downward path starting at the root
- The next node visited depends on the comparison of ***k*** with the key of the current node
- If we reach an external node, the key is not found
- Example: get(4):
  - Call TreeSearch(4,root)
- The algorithms for nearest neighbor queries (predecessor and successor) are similar.

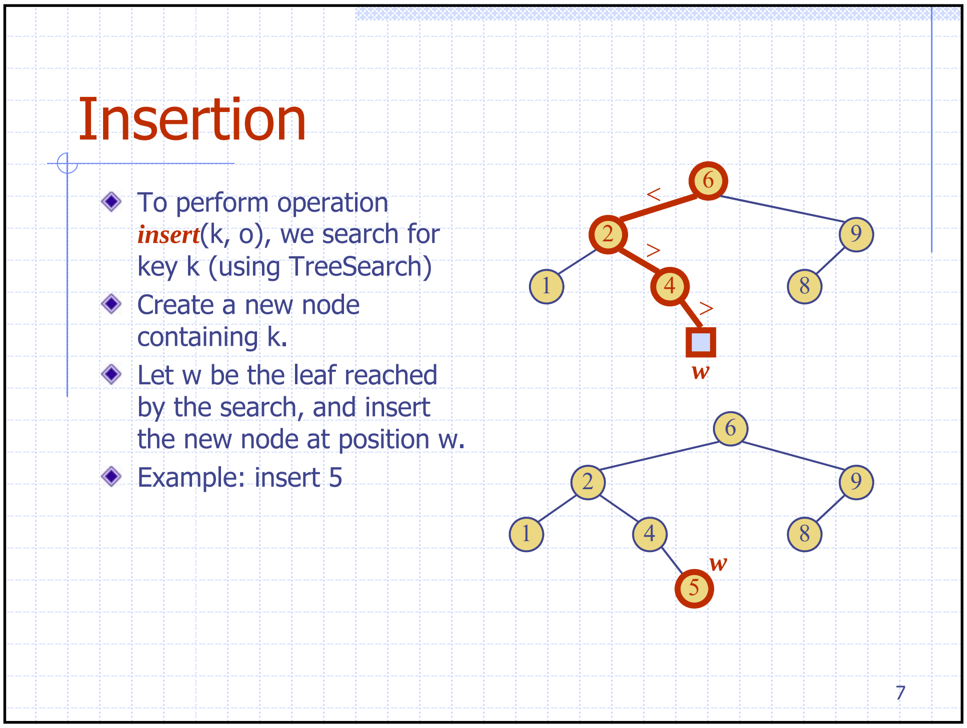
```
Algorithm TreeSearch(k, v)
    if isNull (v)
        return v    // v is null or empty node
    if k < key(v)
        return TreeSearch(k, leftChild(v))
    else if k = key(v)
        return v    // key(v) = k.
    else // k > key(v)
        return TreeSearch(k, rightChild(v))
```



Minimum & Maximum

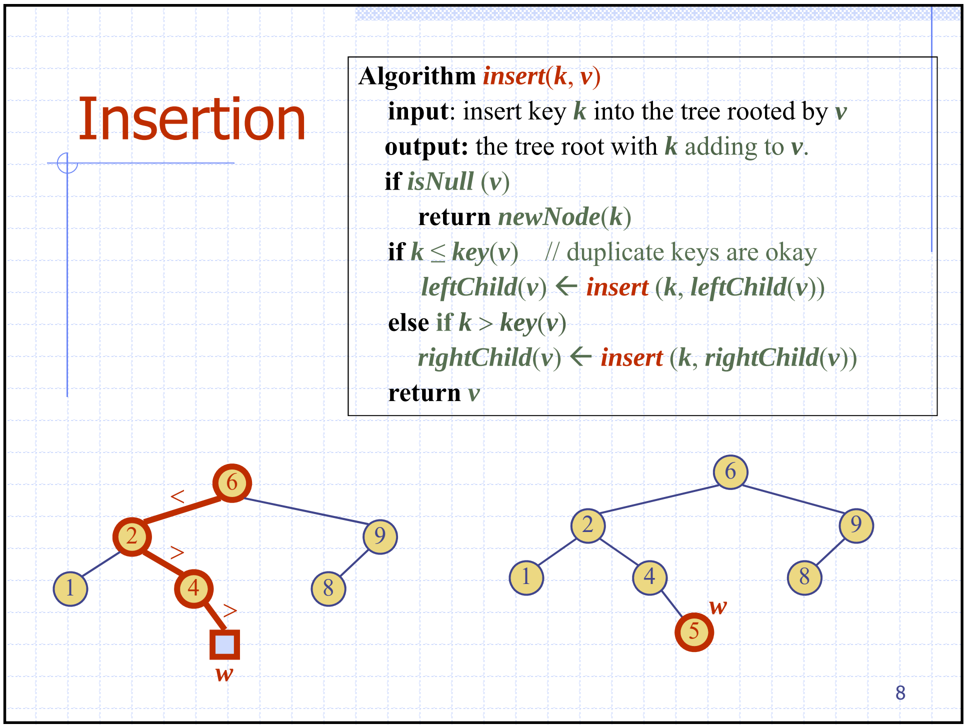
- The minimum node is null if the root is null; otherwise, it is the leftmost node.
- The maximum node is null if the root is null; otherwise, it is the rightmost node.

```
Algorithm TreeMinimum(v)
    if isNull (v)
        return v    // v is null or empty node
    if isNull(leftChild(v))
        return  v
    return TreeMinimum(leftChild(v))
```



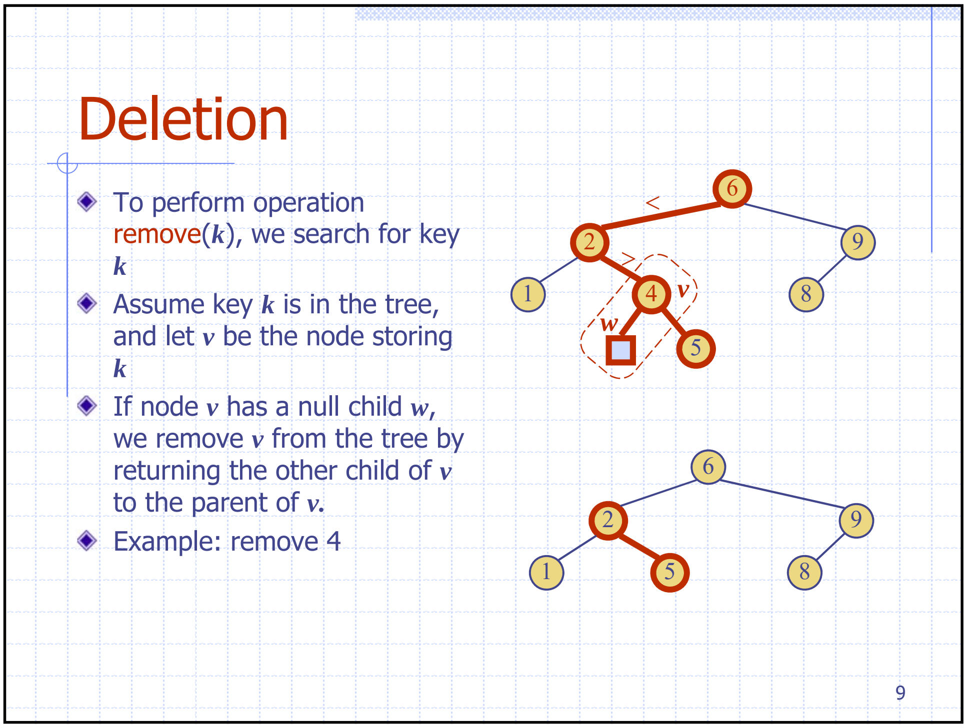
Insertion

- To perform operation ***insert***(k, o), we search for key k (using TreeSearch)
- Create a new node containing k.
- Let w be the leaf reached by the search, and insert the new node at position w.
- Example: insert 5



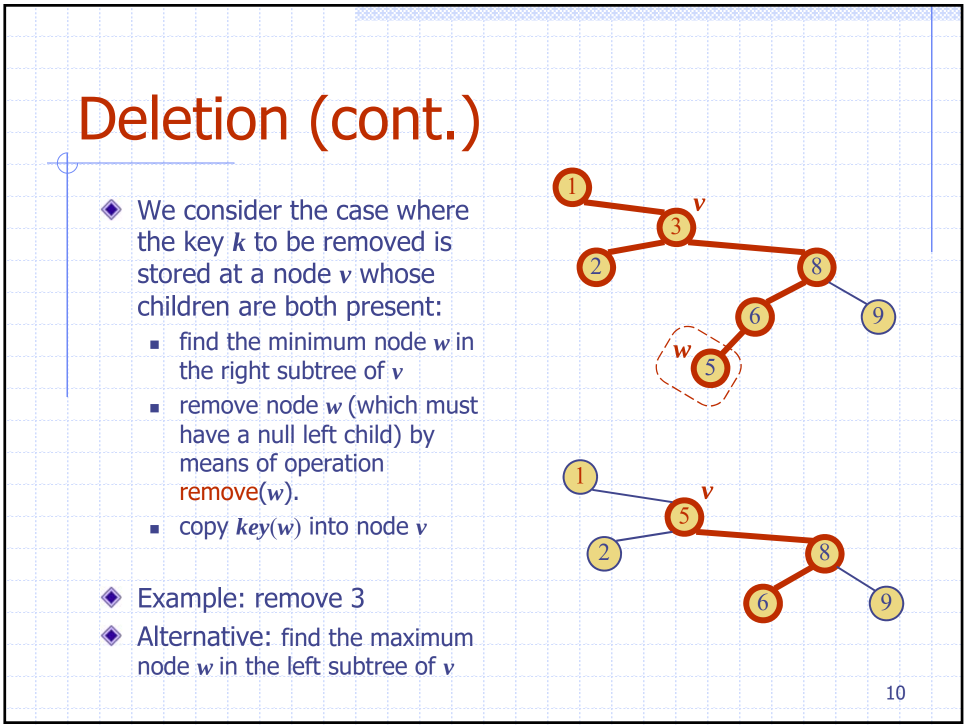
Insertion

```
Algorithm insert(k, v)
    input: insert key k into the tree rooted by v
    output: the tree root with k adding to v.
    if isNull (v)
        return newNode(k)
    if $k \le key(v)$    // duplicate keys are okay
        leftChild(v) ← insert (k, leftChild(v))
    else if $k > key(v)$
        rightChild(v) ← insert (k, rightChild(v))
    return v
```



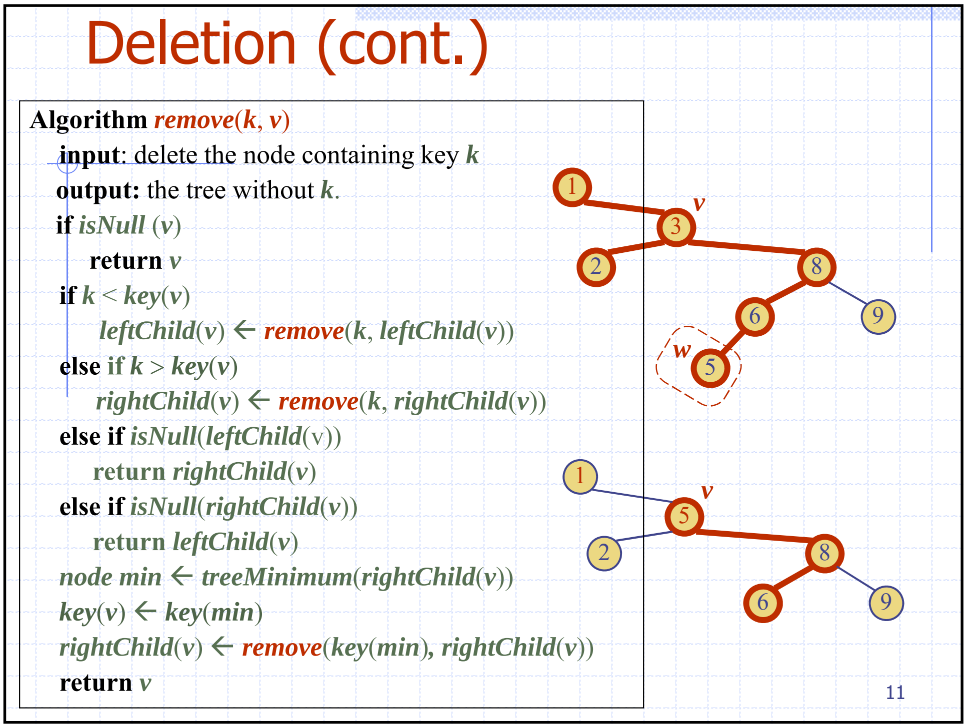
Deletion

- To perform operation remove(*k*), we search for key *k*
- Assume key *k* is in the tree, and let *v* be the node storing *k*
- If node *v* has a null child *w*, we remove *v* from the tree by returning the other child of *v* to the parent of *v*.
- Example: remove 4



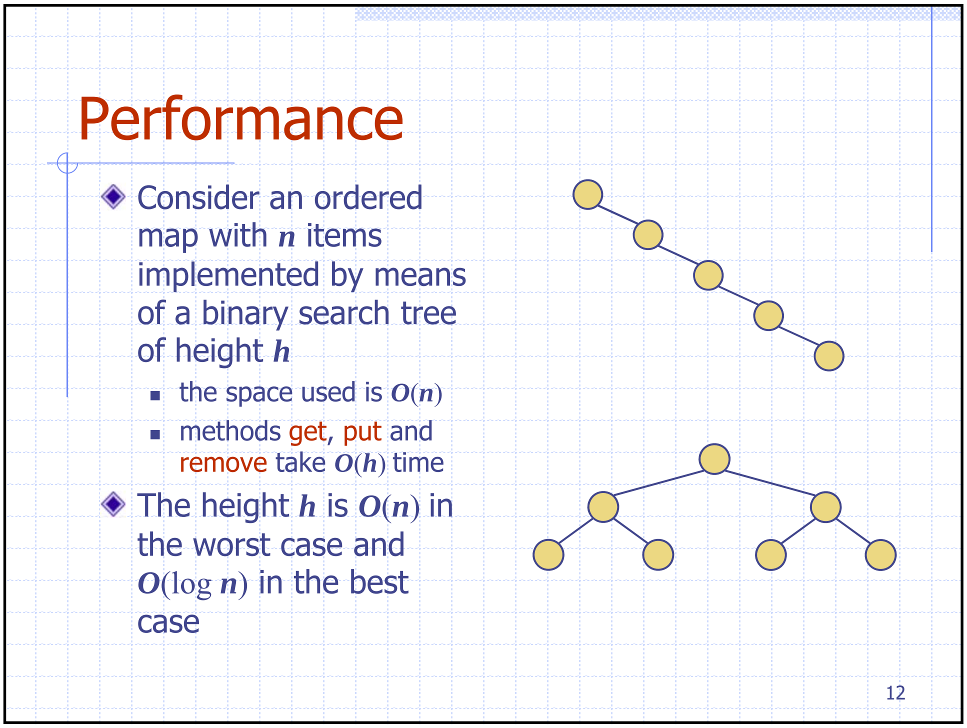
Deletion (cont.)

- We consider the case where the key *k* to be removed is stored at a node *v* whose children are both present:
  - find the minimum node *w* in the right subtree of *v*
  - remove node *w* (which must have a null left child) by means of operation remove(*w*).
  - copy **key**(*w*) into node *v*
- Example: remove 3
- Alternative: find the maximum node *w* in the left subtree of *v*



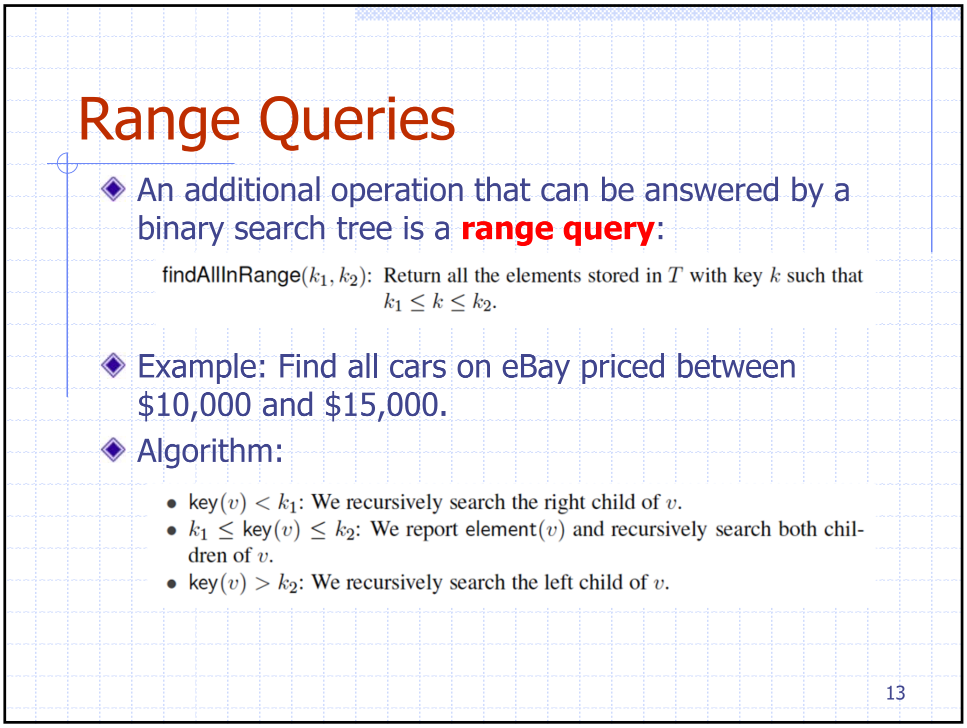
Deletion (cont.)

```
Algorithm remove(k, v)
    input: delete the node containing key k
    output: the tree without k.
    if isNull (v)
        return v
    if k < key(v)
        leftChild(v) ← remove(k, leftChild(v))
    else if k > key(v)
        rightChild(v) ← remove(k, rightChild(v))
    else if isNull(leftChild(v))
        return rightChild(v)
    else if isNull(rightChild(v))
        return leftChild(v)
    node min ← treeMinimum(rightChild(v))
    key(v) ← key(min)
    rightChild(v) ← remove(key(min), rightChild(v))
    return v
```



Performance

- Consider an ordered map with n items implemented by means of a binary search tree of height h
  - the space used is $O(n)$
  - methods get, put and remove take $O(h)$ time
- The height h is $O(n)$ in the worst case and $O(\log n)$ in the best case

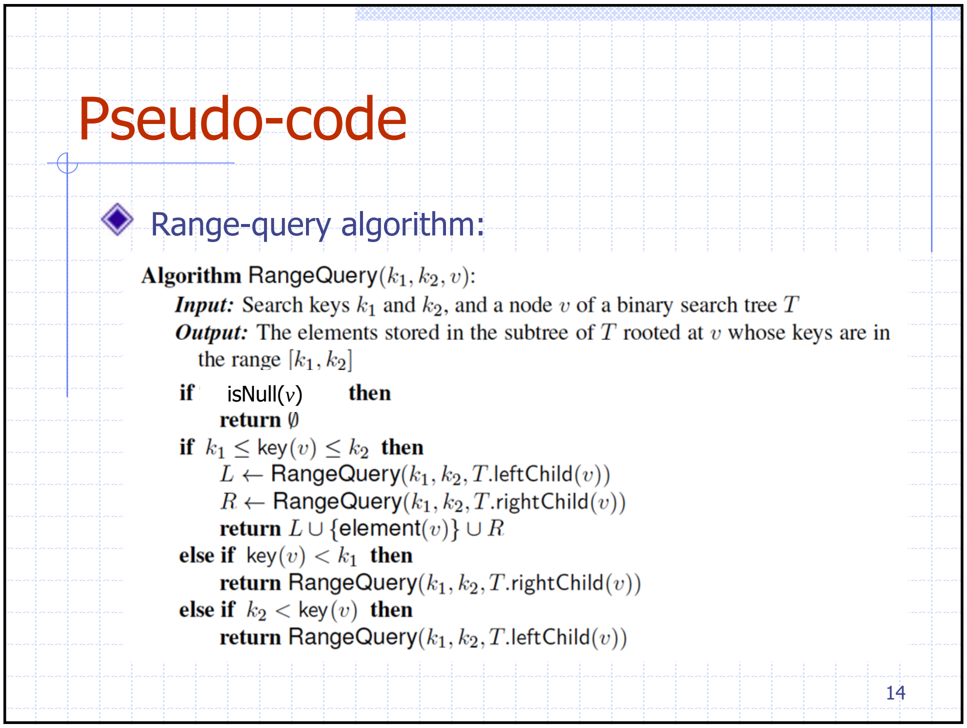


Range Queries

- An additional operation that can be answered by a binary search tree is a **range query**:

findAllInRange(k_1, k_2): Return all the elements stored in T with key k such that $k_1 \leq k \leq k_2$.

- Example: Find all cars on eBay priced between $10,000 and $15,000.
- Algorithm:
  - key$(v) < k_1$: We recursively search the right child of v.
  - $k_1 \leq$ key$(v) \leq k_2$: We report element(v) and recursively search both children of v.
  - key$(v) > k_2$: We recursively search the left child of v.



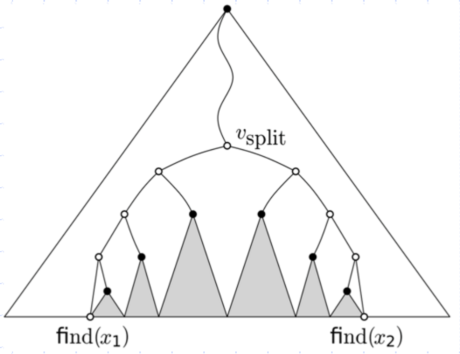
Pseudo-code

- Range-query algorithm:

```
Algorithm RangeQuery($k_1, k_2, v$):
    Input: Search keys $k_1$ and $k_2$, and a node $v$ of a binary search tree $T$
    Output: The elements stored in the subtree of $T$ rooted at $v$ whose keys are in
      the range $[k_1, k_2]$
    if  isNull($v$)  then
        return $\emptyset$
    if $k_1 \leq$ key$(v) \leq k_2$ then
        $L \leftarrow$ RangeQuery$(k_1, k_2, T$.leftChild$(v))$
        $R \leftarrow$ RangeQuery$(k_1, k_2, T$.rightChild$(v))$
        return $L \cup \{$element$(v)\} \cup R$
    else if key$(v) < k_1$ then
        return RangeQuery$(k_1, k_2, T$.rightChild$(v))$
    else if $k_2 <$ key$(v)$ then
        return RangeQuery$(k_1, k_2, T$.leftChild$(v))$
```

Visualization

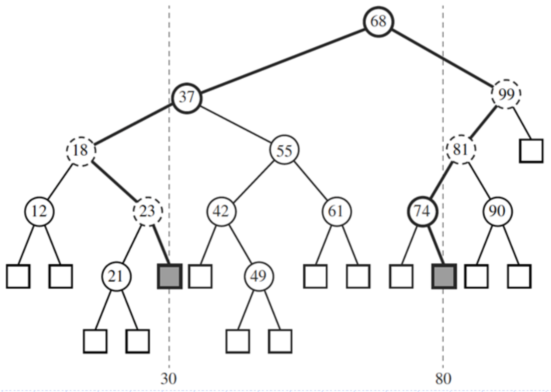
- Drawing subtrees as triangles, then we visit all the shaded subtrees.

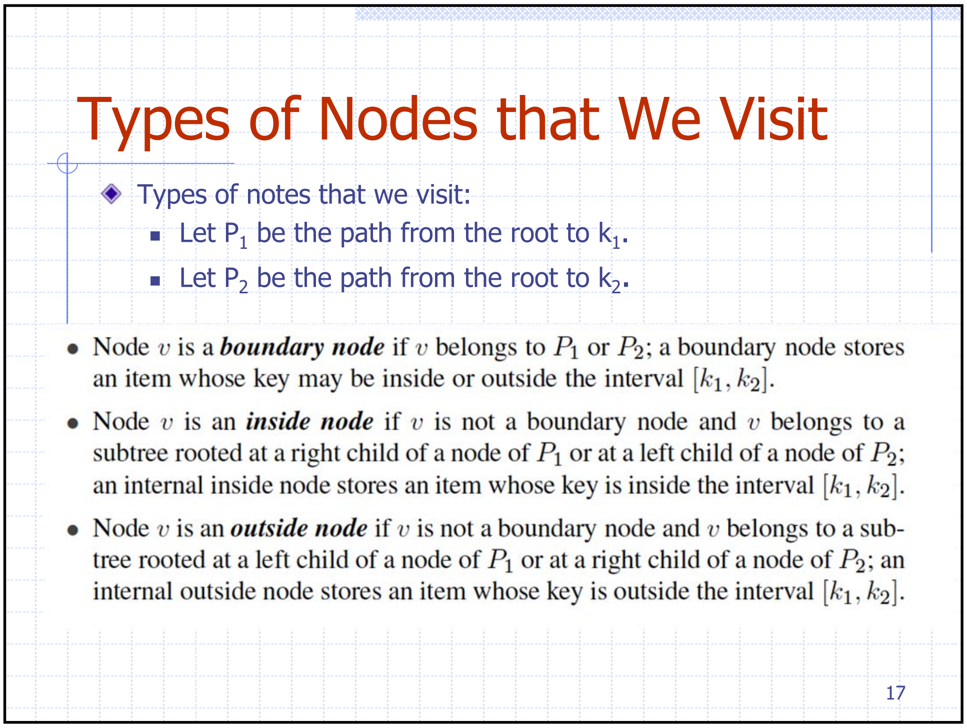


A range query on a 1-dimensional range tree.
Magnus Manske, 2012. Public-domain image.

Example

- An example shows that we also need to test for the nodes we visit along the search paths for k_1 and k_2.

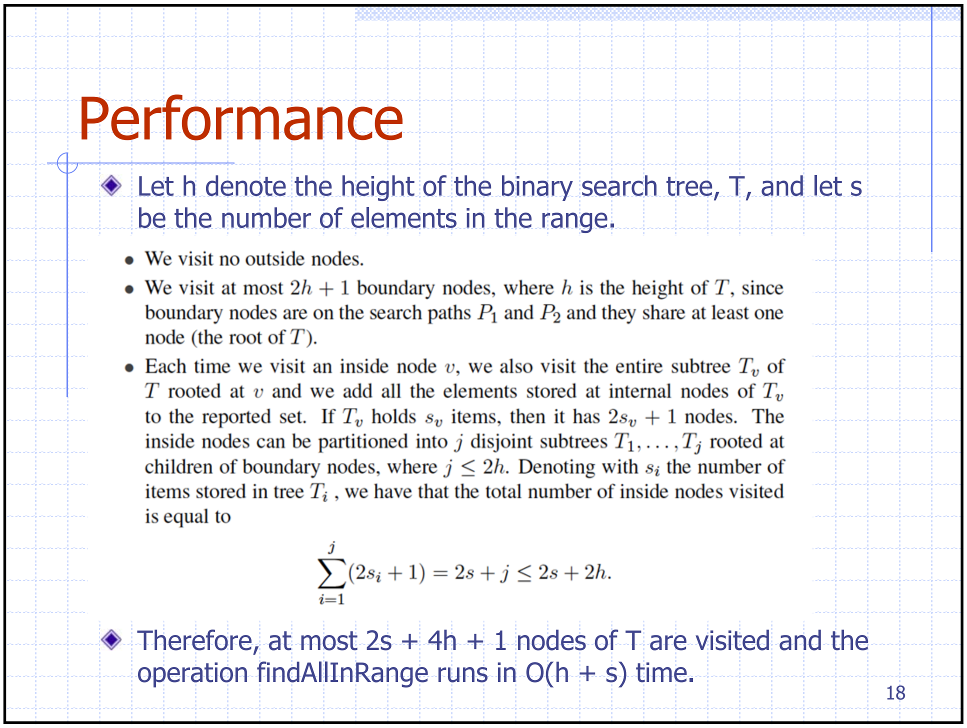




Types of Nodes that We Visit

- Types of notes that we visit:
  - Let P_1 be the path from the root to k_1.
  - Let P_2 be the path from the root to k_2.

- Node v is a ***boundary node*** if v belongs to P_1 or P_2; a boundary node stores an item whose key may be inside or outside the interval $[k_1, k_2]$.
- Node v is an ***inside node*** if v is not a boundary node and v belongs to a subtree rooted at a right child of a node of P_1 or at a left child of a node of P_2; an internal inside node stores an item whose key is inside the interval $[k_1, k_2]$.
- Node v is an ***outside node*** if v is not a boundary node and v belongs to a subtree rooted at a left child of a node of P_1 or at a right child of a node of P_2; an internal outside node stores an item whose key is outside the interval $[k_1, k_2]$.



Performance

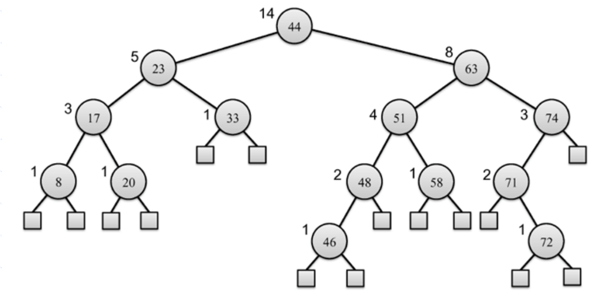
- Let h denote the height of the binary search tree, T, and let s be the number of elements in the range.
  - We visit no outside nodes.
  - We visit at most $2h + 1$ boundary nodes, where h is the height of T, since boundary nodes are on the search paths P_1 and P_2 and they share at least one node (the root of T).
  - Each time we visit an inside node v, we also visit the entire subtree T_v of T rooted at v and we add all the elements stored at internal nodes of T_v to the reported set. If T_v holds s_v items, then it has $2s_v + 1$ nodes. The inside nodes can be partitioned into j disjoint subtrees $T_1, \ldots, T_j$ rooted at children of boundary nodes, where $j \leq 2h$. Denoting with s_i the number of items stored in tree T_i, we have that the total number of inside nodes visited is equal to

$$\sum_{i=1}^{j}(2s_i + 1) = 2s + j \leq 2s + 2h.$$

- Therefore, at most 2s + 4h + 1 nodes of T are visited and the operation findAllInRange runs in O(h + s) time.

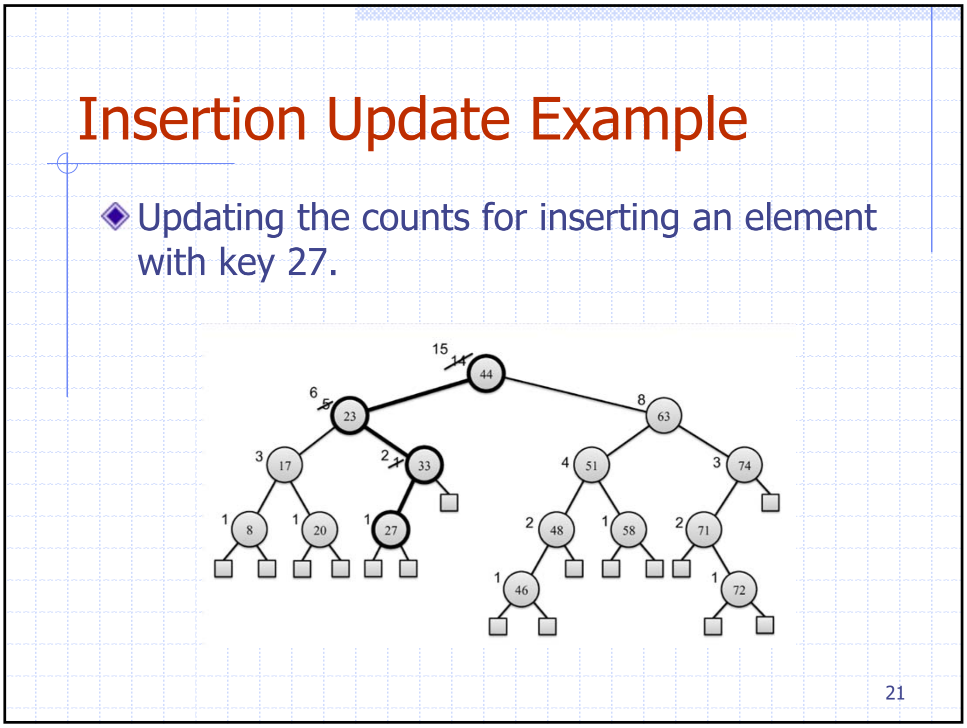
Index-Based Searching (Selection)

- Add a new operation:
  - select(*i*): Return the item with the *i* th smallest key, where 1 ≤ *i* ≤ n.
- Main idea to support this new method:
  - Augment each node v to store n_v, the number of elements in the subtree rooted at v.



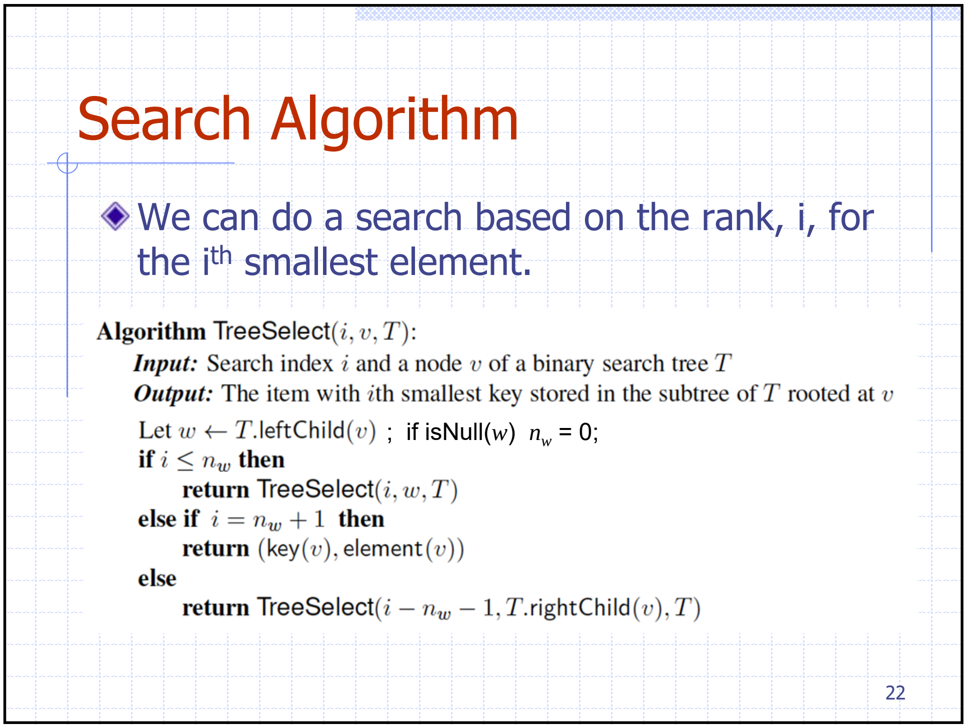
Maintaining the New Fields

- We must now update n_v fields when we do an insertion or deletion.
  - If we are doing an insertion by creating a new node, w, in T, then we set n_w = 1 and we increment the n_v count for each node v that is an ancestor of w, that is, on the path from w to the root of T.
  - If we are doing a deletion at a node, w, in T, then we decrement the n_v count for each node v that is on the path from w's parent to the root of T.



Insertion Update Example

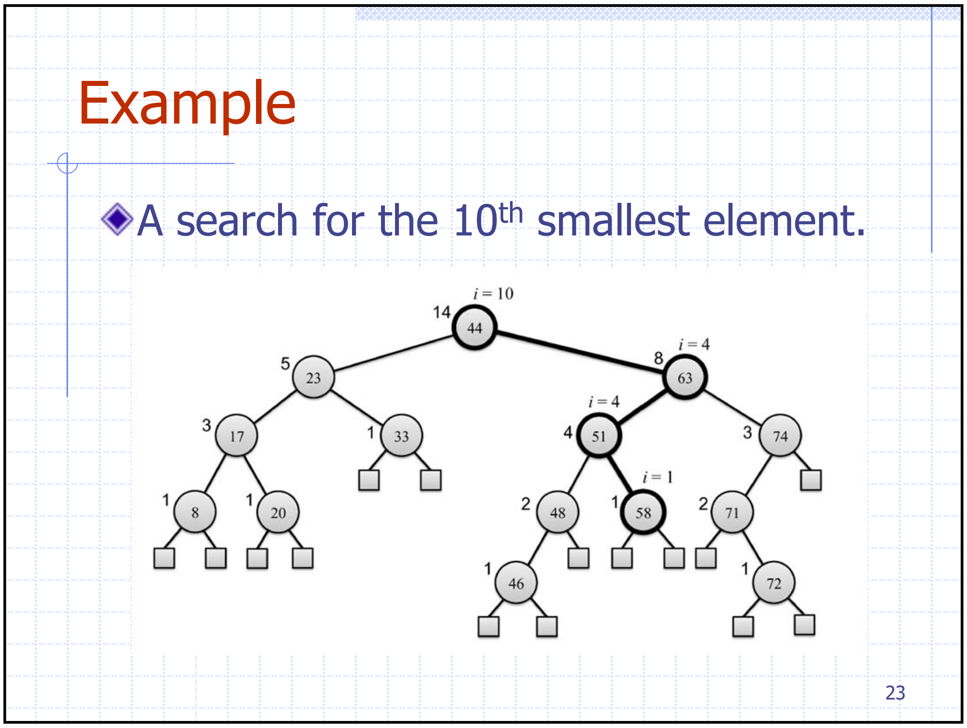
- Updating the counts for inserting an element with key 27.



Search Algorithm

- We can do a search based on the rank, i, for the i^{th} smallest element.

```
Algorithm TreeSelect($i, v, T$):
    Input: Search index $i$ and a node $v$ of a binary search tree $T$
    Output: The item with $i$th smallest key stored in the subtree of $T$ rooted at $v$
    Let $w \leftarrow T$.leftChild($v$) ;  if isNull($w$)  $n_w$ = 0;
    if $i \leq n_w$ then
        return TreeSelect($i, w, T$)
    else if  $i = n_w + 1$  then
        return (key($v$), element($v$))
    else
        return TreeSelect($i - n_w - 1, T$.rightChild($v$), $T$)
```



Example

- A search for the 10th smallest element.