Functions

22c:19
Chapter 7
Hantao Zhang

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Definition of a function

- A function takes an element from a set and maps it to a UNIQUE element in another set

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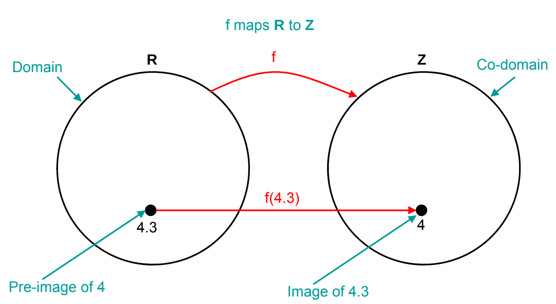
Function terminology: floor



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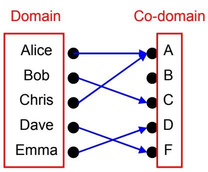
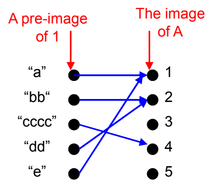
More functions

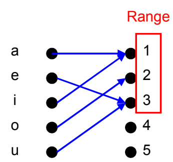


A class grade function

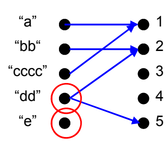


A string length function

Even more functions



Some function…



Not a valid function!
Also not a valid function!

Function arithmetic

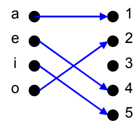
- Let $f_1(x) = 2x$
- Let $f_2(x) = x^2$

- f_1+f_2 is a func: $(f_1+f_2)(x) = f_1(x)+f_2(x) = 2x+x^2$

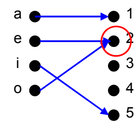
- f_1*f_2 is a func: $(f_1*f_2)(x) = f_1(x)*f_2(x) = 2x*x^2 = 2x^3$

One-to-one functions

- A function is one-to-one if each element in the co-domain has a unique pre-image
  - Meaning no 2 values map to the same result



A one-to-one function



A function that is not one-to-one

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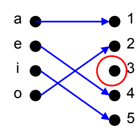
More on one-to-one

- Injective is synonymous with one-to-one
  - "A function is injective"
- A function is injective if it is one-to-one
- Note that there can be un-used elements in the co-domain



A one-to-one function

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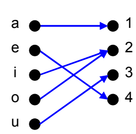
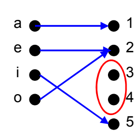
Onto functions

- A function is onto if each element in the co-domain is an image of some pre-image
  - Meaning all elements in the right are mapped to



An onto function



A function that is not onto

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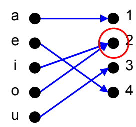
More on onto

- Surjective is synonymous with onto
  - "A function is surjective"
- A function is surjective if it is onto

- Note that there can be multiply used elements in the co-domain



An onto function

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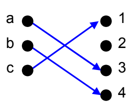
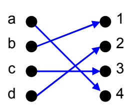
Onto vs. one-to-one

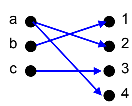
- Are the following functions onto, one-to-one, both, or neither?



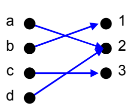
1-to-1, not onto



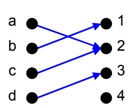
Both 1-to-1 and onto



Not a valid function



Onto, not 1-to-1



Neither 1-to-1 nor onto

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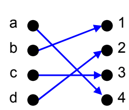
Bijections

- Consider a function that is both one-to-one and onto:



- Such a function is a one-to-one correspondence, or a bijection

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Identity functions

- A function such that the image and the pre-image are ALWAYS equal
- f(x) = 1*x
- f(x) = x + 0
- The domain and the co-domain must be the same set

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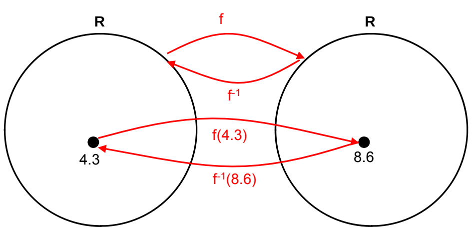
Inverse functions

Let f(x) = 2*x



Then $f^{-1}(x) = x/2$

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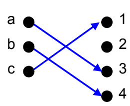
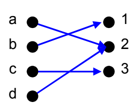
More on inverse functions

- Can we define the inverse of the following functions?



What is $f^{-1}(2)$?
Not onto!



What is $f^{-1}(2)$?
Not 1-to-1!

- An inverse function can ONLY be done defined on a bijection

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Compositions of functions

- Let (f ○ g)(x) = f(g(x))
- Let f(x) = 2x+3 Let g(x) = 3x+2
- g(1) = 5, f(5) = 13
- Thus, (f ○ g)(1) = f(g(1)) = 13

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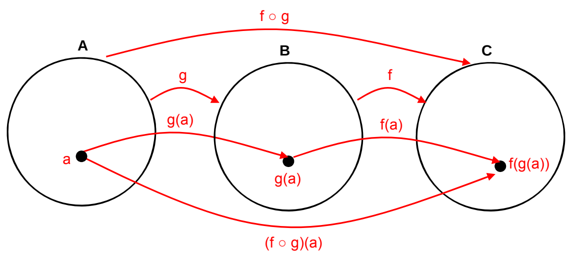
Compositions of functions



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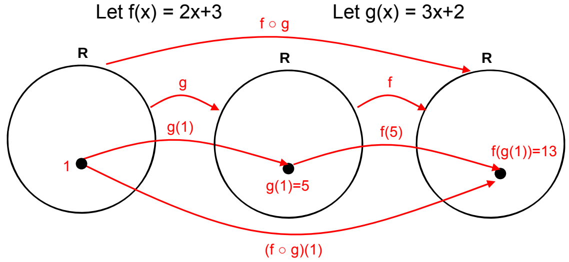
Compositions of functions

Let f(x) = 2x+3 Let g(x) = 3x+2



f(g(x)) = 2(3x+2)+3 = 6x+7

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Compositions of functions

Does f(g(x)) = g(f(x))?

Let f(x) = 2x+3   Let g(x) = 3x+2

f(g(x)) = 2(3x+2)+3 = 6x+7
g(f(x)) = 3(2x+3)+2 = 6x+11

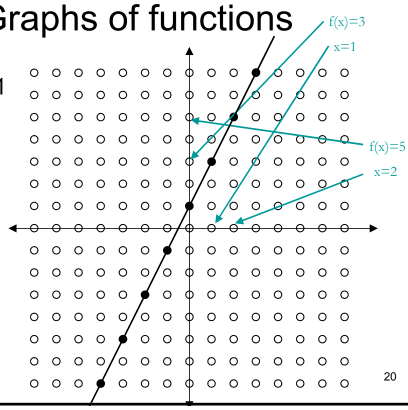
Not equal!

Function composition is not commutative!

Graphs of functions

Let f(x)=2x+1

Plot (x, f(x))



This is a plot of f(x)

Useful functions

- Floor: ⌊x⌋ means take the greatest integer less than or equal to the number
- Ceiling: ⌈x⌉ means take the lowest integer greater than or equal to the number
- round(x) = floor(x+0.5)

Sample floor/ceiling questions

- Find these values

$\lfloor 1.1 \rfloor$	1
$\lceil 1.1 \rceil$	2
$\lfloor -0.1 \rfloor$	-1
$\lceil -0.1 \rceil$	0
$\lceil 2.99 \rceil$	3
$\lceil -2.99 \rceil$	-2
$\lfloor ½+\lceil ½ \rceil \rfloor$	$\lfloor ½+1 \rfloor = \lfloor 3/2 \rfloor = 1$
$\lceil \lfloor ½ \rfloor + \lceil ½ \rceil + ½ \rceil$	$\lceil 0 + 1 + ½ \rceil = \lceil 3/2 \rceil = 2$

Ceiling and floor properties

Let n be an integer

(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$

(1b) $\lceil x \rceil = n$ if and only if $n-1 < x \le n$

(1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$

(1d) $\lceil x \rceil = n$ if and only if $x \le n < x+1$

(2) $x-1 < \lfloor x \rfloor \le x \le = \lceil x \rceil < x+1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x+n \rceil = \lceil x \rceil + n$

Ceiling property proof

- Prove rule 4a: $\lfloor x+n \rfloor = \lfloor x \rfloor + n$
  - Where n is an integer
  - Will use rule 1a: $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
- Direct proof!
  - Let $m = \lfloor x \rfloor$
  - Thus, $m \le x < m+1$ (by rule 1a)
  - Add n to both sides: $m+n \le x+n < m+n+1$
  - By rule 4a, $m+n = \lfloor x+n \rfloor$
  - Since $m = \lfloor x \rfloor$, m+n also equals $\lfloor x \rfloor + n$
  - Thus, $\lfloor x \rfloor + n = m+n = \lfloor x+n \rfloor$

Proving function problems

- Let *f* be a function from A to B, and let S and T be subsets of A. We use f(S) to denote the set { f(x) | x ∈ S}, f(T) to denote the set { f(x) | x ∈ T}, and f(S U T) to denote the set { f(x) | x ∈ S U T}.
- Example: f is the floor function.
  f({5.1, 5.7, 6.2, 6.4}) = { 5, 6 }.
- Show that

$$a)\ f(S \cup T) = f(S) \cup f(T)$$
$$b)\ f(S \cap T) \subseteq f(S) \cap f(T)$$

Proving function problems

- f(SUT) = f(S) U f(T)
- Will show that each side is a subset of the other
- Two cases!
- Case1: Show that f(SUT) ⊆ f(S) U f(T)
  - Let b ∈ f(SUT). Thus, b=f(a) for some a∈S U T
  - Either a∈S, in which case b∈f(S)
  - Or a∈T, in which case b∈f(T)
  - Thus, b∈f(S) U f(T)
- Case 2: Show that f(S) U f(T) ⊆ f(S U T)
  - Let b ∈ f(S) U f(T)
  - Either b ∈ f(S) or b ∈ f(T) (or both!)
  - Thus, b = f(a) for some a ∈ S or some a ∈ T
  - In either case, b = f(a) for some a ∈ S U T

Proving function problems

- f(S∩T) ⊆ f(S) ∩ f(T)

- Let b ∈ f(S∩T). Then b = f(a) for some a ∈ S∩T
- This implies that a ∈ S and a ∈ T
- Thus, b ∈ f(S) and b ∈ f(T)
- Therefore, b ∈ f(S) ∩ f(T)

- It is NOT the case that f(S)∩f(T) ⊆ f(S∩T)
- Let f(x) = 1. S = {1}, T = {2}.
- Then f(S) = {1} = f(T), S∩T = empty.
- f(S)∩f(T) = {1} but f(S∩T) = empty.

Proving function problems

- Let *f* be an invertible function from Y to Z
- Both $f \circ f^{-1}$ and $f^{-1} \circ f$ are identity functions:
- $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

- Let *g* be also an invertible function from X to Y
- We show that the inverse of $f \circ g$ is:
  - $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

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Proving function problems

- Thus, we want to show, for all $z \in Z$ and $x \in X$

$$((f \circ g) \circ (g^{-1} \circ f^{-1}))(z) = z$$
$$((g^{-1} \circ f^{-1}) \circ (f \circ g))(x) = x$$

$$\begin{aligned}((f \circ g) \circ (g^{-1} \circ f^{-1}))(z) &= (f \circ g)((g^{-1} \circ f^{-1})(z)) \\ &= (f \circ g)(g^{-1}(f^{-1}(z))) \\ &= f(g(g^{-1}(f^{-1}(z)))) \\ &= f(f^{-1}(z)) \\ &= z\end{aligned}$$

- The second equality is similar

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The pigeonhole principle

- Suppose a flock of pigeons fly into a set of pigeonholes to roost

- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole that has more than one pigeon in it

- If *k*+1 or more objects are placed into *k* boxes, then there is at least one box containing two or more of the objects
  - This is Theorem 1

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Pigeonhole principle examples

- In a group of 367 people, there must be two people with the same birthday
  - As there are 366 possible birthdays
- In a group of 27 English words, at least two words must start with the same letter
  - As there are only 26 letters

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Generalized pigeonhole principle

- If *N* objects are placed into *k* boxes, then there is at least one box containing $\lceil N/k \rceil$ objects
  - This is Theorem 2

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Generalized pigeonhole principle examples

- Among 100 people, there are at least $\lceil 100/12 \rceil$ = 9 born on the same month
- How many students in a class must there be to ensure that 6 students get the same grade (one of A, B, C, D, or F)?
  - The "boxes" are the grades. Thus, $k = 5$
  - Thus, we set $\lceil N/5 \rceil = 6$
  - Lowest possible value for *N* is 26

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Sample questions

- A bowl contains 10 red and 10 yellow balls

a) How many balls must be selected to ensure 3 balls of the same color?
   - One solution: consider the “worst” case
     - Consider 2 balls of each color
     - You can't take another ball without hitting 3
     - Thus, the answer is 5
   - Via generalized pigeonhole principle
     - How many balls are required if there are 2 colors, and one color must have 3 balls?
     - How many pigeons are required if there are 2 pigeon holes, and one must have 3 pigeons?
     - number of boxes: $k = 2$
     - We want $\lceil N/k \rceil = 3$
     - What is the minimum N?
     - $N = 5$

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Sample questions

- A bowl contains 10 red and 10 yellow balls

b) How many balls must be selected to ensure 3 yellow balls?
   - Consider the “worst” case
     - Consider 10 red balls and 2 yellow balls
     - You can't take another ball without hitting 3 yellow balls
     - Thus, the answer is 13

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Sample questions

- 6 computers on a network are connected to at least 1 other computer
- Show there are at least two computers that are have the same number of connections

- The number of boxes, k, is the number of computer connections
  - This can be 1, 2, 3, 4, or 5
- The number of pigeons, N, is the number of computers
  - That's 6
- By the generalized pigeonhole principle, at least one box must have $\lceil N/k \rceil$ objects
  - $\lceil 6/5 \rceil = 2$
  - In other words, at least two computers must have the same number of connections

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How many functions?

- Let f be a function from A to B, |A| = m, |B| = n.
- How many different f?
- Each x in A has n choices for its image.
- Total =n^m
- How many different one-to-one f?
- Must m <= n. First x in A has n choices, second has (n-1) choices, ….
- Total = P(n, m) = n!/(n-m)!.
- How many different bijective f?
- Must m = n.
- Each bijective f is a permutation of A: P(n, n) = n!.
- How many different onto f?

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How many functions?

- Let f be a function from A to B, |A| = m, |B| = n.
- How many different onto f?
- Must m >= n.
- We have m! ways to arrange elements of A in a list.
- For each list, we partition the list into n non-empty sublists: each sublist is the pre-images of a element of B.
- We just need to mark which element is the first in a sublist.
- The first element in the first sublist is fixed.
- We need to choose n-1 slots out of m-1 slots in the list. That is C(m-1, n-1) = (m-1)!/((m-n)!(n-1)!)
- Total = m!C(m-1, n-1).
- Ex: m = 5, n = 4, there are 5!C(4,3) = 120*4 = 480 onto f.

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