

**Introduction to Discrete
Probability**

22c:19, section 6.x
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1

Terminology

- Experiment
 - A repeatable procedure that yields one of a given set of outcomes
 - Rolling a die, for example
- Sample space
 - The range of outcomes possible
 - For a die, that would be values 1 to 6
- Event
 - One of the sample outcomes that occurred
 - If you rolled a 4 on the die, the event is the 4

2

Probability definition

- The probability of an event occurring is:
$$p(E) = \frac{|E|}{|S|}$$
 - Where E is the set of desired events (outcomes) and
 - S is the set of all possible events (outcomes)
 - Note that $0 \leq |E| \leq |S|$
 - Thus, the probability will always be between 0 and 1
 - An event that will never happen has probability 0
 - An event that will always happen has probability 1

3

Probability is always a value between 0 and 1

- Something with a probability of 0 will never occur
- Something with a probability of 1 will always occur
- You cannot have a probability outside this range!
- Note that when somebody says it has a “100% probability)
 - That means it has a probability of 1

4

Dice probability

- What is the probability of getting “snake-eyes” (two 1’s) on two six-sided dice?
 - Probability of getting a 1 on a 6-sided die is $1/6$
 - Via product rule, probability of getting two 1’s is the probability of getting a 1 AND the probability of getting a second 1
 - Thus, it’s $1/6 * 1/6 = 1/36$
- What is the probability of getting a 7 by rolling two dice?
 - There are six combinations that can yield 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)
 - Thus, $|E| = 6, |S| = 36, P(E) = 6/36 = 1/6$

5

The game of poker

- You are given 5 cards (this is 5-card stud poker)
- The goal is to obtain the best hand you can
- The possible poker hands are (in increasing order):
 - No pair
 - One pair (two cards of the same face)
 - Two pair (two sets of two cards of the same face)
 - Three of a kind (three cards of the same face)
 - Straight (all five cards sequentially – ace is either high or low)
 - Flush (all five cards of the same suit)
 - Full house (a three of a kind of one face and a pair of another face)
 - Four of a kind (four cards of the same face)
 - Straight flush (both a straight and a flush)
 - Royal flush (a straight flush that is 10, J, K, Q, A)

6

Poker probability: royal flush

- What is the chance of getting a royal flush?
 - That's the cards 10, J, Q, K, and A of the same suit
- There are only 4 possible royal flushes
- Possibilities for 5 cards: $C(52,5) = 2,598,960$
- Probability = $4/2,598,960 = 0.0000015$
 - Or about 1 in 650,000



7

Poker probability: four of a kind

- What is the chance of getting 4 of a kind when dealt 5 cards?
 - Possibilities for 5 cards: $C(52,5) = 2,598,960$
- Possible hands that have four of a kind:
 - There are 13 possible four of a kind hands
 - The fifth card can be any of the remaining 48 cards
 - Thus, total possibilities is $13 \cdot 48 = 624$
- Probability = $624/2,598,960 = 0.00024$
 - Or 1 in 4165

8

Poker probability: flush

- What is the chance of getting a flush?
 - That's all 5 cards of the same suit
- We must do ALL of the following:
 - Pick the suit for the flush: $C(4,1)$
 - Pick the 5 cards in that suit: $C(13,5)$



- As we must do all of these, we multiply the values out (via the product rule)
- This yields $\binom{13}{5} \binom{4}{1} = 5148$
- Possibilities for 5 cards: $C(52,5) = 2,598,960$
- Probability = $5148/2,598,960 = 0.00198$
 - Or about 1 in 505
- Note that if you don't count straight flushes as "flush" or "royal flush", then the number is really 5108

Poker probability: full house

- What is the chance of getting a full house?
 - That's three cards of one face and two of another face
- We must do ALL of the following:
 - Pick the face for the three of a kind: $C(13,1)$
 - Pick the 3 of the 4 cards to be used: $C(4,3)$
 - Pick the face for the pair: $C(12,1)$
 - Pick the 2 of the 4 cards of the pair: $C(4,2)$
- As we must do all of these, we multiply the values out (via the product rule)
- This yields $\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2} = 3744$
- Possibilities for 5 cards: $C(52,5) = 2,598,960$
- Probability = $3744/2,598,960 = 0.00144$
 - Or about 1 in 694



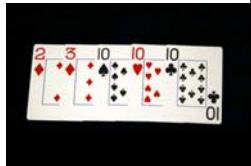
Inclusion-exclusion principle

- The possible poker hands are (in increasing order):
 - Nothing
 - One pair cannot include two pair, three of a kind, four of a kind, or full house
 - Two pair cannot include three of a kind, four of a kind, or full house
 - Three of a kind cannot include four of a kind or full house
 - Straight cannot include straight flush or royal flush
 - Flush cannot include straight flush or royal flush
 - Full house
 - Four of a kind
 - Straight flush cannot include royal flush
 - Royal flush

11

Poker probability: three of a kind

- What is the chance of getting a three of a kind?
 - That's three cards of one face
 - Can't include a full house or four of a kind
- We must do ALL of the following:
 - Pick the face for the three of a kind: $C(13,1)$
 - Pick the 3 of the 4 cards to be used: $C(4,3)$
 - Pick the two other cards' face values: $C(12,2)$
 - We can't pick two cards of the same face!
 - Pick the suits for the two other cards: $C(4,1) \cdot C(4,1)$
- As we must do all of these, we multiply the values out (via the product rule)
- This yields $\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} = 54912$
- Possibilities for 5 cards: $C(52,5) = 2,598,960$
- Probability = $54,912/2,598,960 = 0.0211$
 - Or about 1 in 47



12

Poker hand odds

- The possible poker hands are (in increasing order):

– Nothing	1,302,540	0.5012
– One pair	1,098,240	0.4226
– Two pair	123,552	0.0475
– Three of a kind	54,912	0.0211
– Straight	10,200	0.00392
– Flush	5,108	0.00197
– Full house	3,744	0.00144
– Four of a kind	624	0.000240
– Straight flush	36	0.0000139
– Royal flush	4	0.00000154

More on probabilities

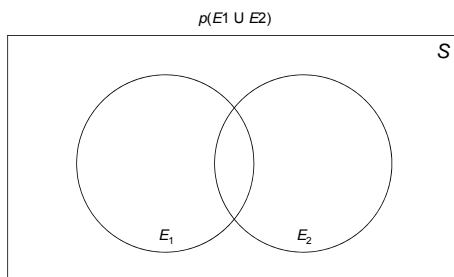
- Let E be an event in a sample space S . The probability of the complement of E is:

$$p(\bar{E}) = 1 - p(E)$$
- Recall the probability for getting a royal flush is 0.0000015
 - The probability of *not* getting a royal flush is $1 - 0.0000015$ or 0.9999985
- Recall the probability for getting a four of a kind is 0.00024
 - The probability of *not* getting a four of a kind is $1 - 0.00024$ or 0.99976

Probability of the union of two events

- Let E_1 and E_2 be events in sample space S
- Then $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
- Consider a Venn diagram dart-board

Probability of the union of two events



16

Probability of the union of two events

- If you choose a number between 1 and 100, what is the probability that it is divisible by 2 or 5 or both?
- Let n be the number chosen
 - $p(2|n) = 50/100$ (all the even numbers)
 - $p(5|n) = 20/100$
 - $p(2|n)$ and $p(5|n) = p(10|n) = 10/100$
 - $p(2|n)$ or $p(5|n) = p(2|n) + p(5|n) - p(10|n)$
 $= 50/100 + 20/100 - 10/100$
 $= 3/5$

17

When is gambling worth it?

- This is a *statistical* analysis, not a moral/ethical discussion
- What if you gamble \$1, and have a $1/2$ probability to win \$10?
 - If you play 100 times, you will win (on average) 50 of those times
 - Each play costs \$1, each win yields \$10
 - For \$100 spent, you win (on average) \$500
 - Average win is \$5 (or $\$10 \cdot 1/2$) per play for every \$1 spent
- What if you gamble \$1 and have a $1/100$ probability to win \$10?
 - If you play 100 times, you will win (on average) 1 of those times
 - Each play costs \$1, each win yields \$10
 - For \$100 spent, you win (on average) \$10
 - Average win is \$0.10 (or $\$10 \cdot 1/100$) for every \$1 spent
- One way to determine if gambling is worth it:
 - probability of winning * payout \geq amount spent
 - Or $p(\text{winning}) \cdot \text{payout} \geq \text{investment}$
 - Of course, this is a *statistical* measure

18

When is lotto worth it?

- Many older lotto games you have to choose 6 numbers from 1 to 48
 - Total possible choices is $C(48,6) = 12,271,512$
 - Total possible winning numbers is $C(6,6) = 1$
 - Probability of winning is 0.0000000814
 - Or 1 in 12.3 million
- If you invest \$1 per ticket, it is only statistically worth it if the payout is > \$12.3 million
 - As, on the “average” you will only make money that way
 - Of course, “average” will require trillions of lotto plays...

19

Powerball lottery

- Modern powerball lottery is a bit different
 - Source: <http://en.wikipedia.org/wiki/Powerball>
- You pick 5 numbers from 1-55
 - Total possibilities: $C(55,5) = 3,478,761$
- You then pick one number from 1-42 (the powerball)
 - Total possibilities: $C(42,1) = 42$
- By the product rule, you need to do both
 - So the total possibilities is $3,478,761 * 42 = 146,107,962$
- While there are many “sub” prizes, the probability for the jackpot is about 1 in 146 million
 - You will “break even” if the jackpot is \$146M
 - Thus, one should only play if the jackpot is greater than \$146M
- If you count in the other prizes, then you will “break even” if the jackpot is \$121M

20

Blackjack

- You are initially dealt two cards
 - 10, J, Q and K all count as 10
 - Ace is EITHER 1 or 11 (player's choice)
- You can opt to receive more cards (a “hit”)
- You want to get as close to 21 as you can
 - If you go over, you lose (a “bust”)
- You play against the house
 - If the house has a higher score than you, then you lose



Blackjack table



22

Blackjack probabilities

- Getting 21 on the first two cards is called a blackjack
 - Or a “natural 21”
- Assume there is only **1 deck of cards**
- Possible blackjack hands:
 - First card is an A, second card is a 10, J, Q, or K
 - $4/52$ for Ace, $16/51$ for the ten card
 - $= (4 \cdot 16)/(52 \cdot 51) = 0.0241$ (or about 1 in 41)
 - First card is a 10, J, Q, or K; second card is an A
 - $16/52$ for the ten card, $4/51$ for Ace
 - $= (16 \cdot 4)/(52 \cdot 51) = 0.0241$ (or about 1 in 41)
- Total chance of getting a blackjack is the sum of the two:
 - $p = 0.0483$, or about 1 in 20.72

23

Blackjack probabilities

- Another way to get 1 in 20.72.
- There are $C(52,2) = 1,326$ possible initial blackjack hands
- Possible blackjack hands:
 - Pick your Ace: $C(4,1)$
 - Pick your 10 card: $C(16,1)$
 - Total possibilities is the product of the two (64)
- Probability is $64/1,326 = 1$ in 20.72 (0.0483)

24

Blackjack probabilities

- Getting 21 on the first two cards is called a blackjack
- Assume there is **an infinite deck of cards**
 - So many that the probably of getting a given card is not affected by any cards on the table
- Possible blackjack hands:
 - First card is an A, second card is a 10, J, Q, or K
 - 4/52 for Ace, 16/52 for second part
 - = $(4 \cdot 16) / (52 \cdot 52) = 0.0236$ (or about 1 in 42)
 - First card is a 10, J, Q, or K; second card is an A
 - 16/52 for first part, 4/52 for Ace
 - = $(16 \cdot 4) / (52 \cdot 52) = 0.0236$ (or about 1 in 42)
- Total chance of getting a blackjack is the sum:
 - $p = 0.0473$, or about 1 in 21
 - More specifically, it's 1 in 21.13 (vs. 20.72)
- In reality, most casinos use "shoes" of 6-8 decks for this reason
 - It slightly lowers the player's chances of getting a blackjack
 - And prevents people from counting the cards...

25

Counting cards and Continuous Shuffling Machines (CSMs)

- Counting cards means keeping track of which cards have been dealt, and how that modifies the chances
 - There are "easy" ways to do this – count all aces and 10-cards instead of all cards
- Yet another way for casinos to get the upper hand
 - It prevents people from counting the "shoes" of 6-8 decks of cards
- After cards are discarded, they are added to the continuous shuffling machine
- Many blackjack players refuse to play at a casino with one
 - So they aren't used as much as casinos would like



26

So always use a single deck, right?

- Most people think that a single-deck blackjack table is better, as the player's odds increase
 - And you can try to count the cards
- But it's usually not the case!
- Normal rules have a 3:2 payout for a blackjack
 - If you bet \$100, you get your \$100 back plus $3/2 \cdot \$100$, or \$150 additional
- Most single-deck tables have a 6:5 payout
 - You get your \$100 back plus $6/5 \cdot \$100$ or \$120 additional
 - This lowered benefit of awards **OUTWEIGHS** the benefit of the single deck!
 - You cannot win money on a 6:5 blackjack table that uses 1 deck
 - Remember, the house always wins

27

Blackjack probabilities: when to hold

- House usually holds on a 17
 - What is the chance of a bust if you draw on a 17? 16? 15?
- Assume all cards have equal probability
- Bust on a draw on a 18
 - 4 or above will bust: that's 10 (of 13) cards that will bust
 - $10/13 = 0.769$ probability to bust
- Bust on a draw on a 17
 - 5 or above will bust: $9/13 = 0.692$ probability to bust
- Bust on a draw on a 16
 - 6 or above will bust: $8/13 = 0.615$ probability to bust
- Bust on a draw on a 15
 - 7 or above will bust: $7/13 = 0.538$ probability to bust
- Bust on a draw on a 14
 - 8 or above will bust: $6/13 = 0.462$ probability to bust

28

Why counting cards doesn't work well...

- If you make two or three mistakes an hour, you lose any advantage
 - And, in fact, cause a disadvantage!
- You lose lots of money learning to count cards
- Then, once you can do so, you are banned from the casinos

29

So why is Blackjack so popular?

- Although the casino has the upper hand, the odds are much closer to 50-50 than with other games
 - Notable exceptions are games that you are not playing against the house – i.e., poker
 - You pay a fixed amount per hand

30

Roulette

- A wheel with 38 spots is spun
 - Spots are numbered 1-36, 0, and 00
 - European casinos don't have the 00
- A ball drops into one of the 38 spots
- A bet is placed as to which spot or spots the ball will fall into
 - Money is then paid out if the ball lands in the spot(s) you bet upon



The Roulette table



The Roulette table

- Bets can be placed on:

– A single number	1/38
– Two numbers	2/38
– Four numbers	4/38
– All even numbers	18/38
– All odd numbers	18/38
– The first 18 nums	18/38
– Red numbers	18/38



The Roulette table

- | Bets can be placed on: | Probability: | Payout: |
|------------------------|--------------|---------|
| – A single number | 1/38 | 36x |
| – Two numbers | 2/38 | 18x |
| – Four numbers | 4/38 | 9x |
| – All even numbers | 18/38 | 2x |
| – All odd numbers | 18/38 | 2x |
| – The first 18 num | 18/38 | 2x |
| – Red numbers | 18/38 | 2x |

34

Roulette

- It has been proven that no advantageous strategies exist
- Including:
 - Learning the wheel's biases
 - Casino's regularly balance their Roulette wheels
 - Using lasers (yes, lasers) to check the wheel's spin
 - What casino will let you set up a laser inside to beat the house?

35

Roulette

- It has been proven that no advantageous strategies exist
- Including:
 - Martingale betting strategy
 - Where you double your bet each time (thus making up for all previous losses)
 - It still won't work!
 - You can't double your money forever
 - It could easily take 50 times to achieve finally win
 - If you start with \$1, then you must put in $\$1 \cdot 2^{50} = \$1,125,899,906,842,624$ to win this way!
 - That's 1 [quadrillion](#)
 - See [http://en.wikipedia.org/wiki/Martingale_\(roulette_system\)](http://en.wikipedia.org/wiki/Martingale_(roulette_system)) for more info

36

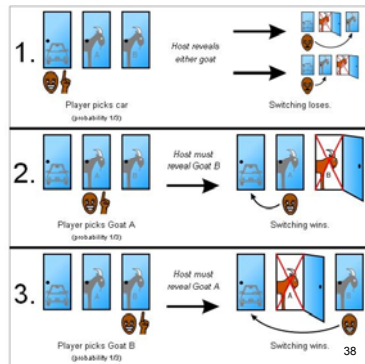
What's behind door number three?

- The Monty Hall problem paradox
 - Consider a game show where a prize (a car) is behind one of three doors
 - The other two doors do not have prizes (goats instead)
 - After picking one of the doors, the host (Monty Hall) opens a different door to show you that the door he opened is not the prize
 - Do you change your decision?
- Your initial probability to win (i.e. pick the right door) is $1/3$
- What is your chance of winning if you change your choice after Monty opens a wrong door?
- After Monty opens a wrong door, if you change your choice, your chance of **winning** is $2/3$
 - Thus, your chance of winning **doubles** if you change
 - Huh?

What's behind door number three?

www.nedmagig.on

Player has equal chance to choose one of the doors. There are three cases.



An aside: probability of multiple events

- Assume you have a $5/6$ chance for an event to happen
 - Rolling a 1-5 on a die, for example
- What's the chance of that event happening twice in a row?
- Cases:
 - Event happening neither time: $1/6 * 1/6 = 1/36$
 - Event happening first time: $5/6 * 1/6 = 5/36$
 - Event happening second time: $1/6 * 5/6 = 5/36$
 - Event happening both times: $5/6 * 5/6 = 25/36$
- For an event to happen twice, the probability is the **product** of the individual probabilities

An aside: probability of multiple events

- Assume you have a $5/6$ chance for an event to happen
 - Rolling a 1-5 on a die, for example
- What's the chance of that event happening **at least once**?
- Cases:
 - Event happening neither time: $1/6 * 1/6 = 1/36$
 - Event happening first time: $5/6 * 1/6 = 5/36$
 - Event happening second time: $1/6 * 5/6 = 5/36$
 - Event happening both times: $5/6 * 5/6 = 25/36$
- It's $35/36$!
- For an event to happen at least once, it's **1 minus the probability of it never happening**
- Or the complement of it never happening

40

Probability vs. odds

- Consider an event that has a 1 in 3 chance of happening
- Probability is 0.333
- Which is a 1 in 3 chance
- Or 2:1 odds
 - Meaning if you play it 3 (2+1) times, you will lose 2 times for every 1 time you win
- This, if you have $x:y$ odds, your probability is $y/(x+y)$
 - The y is usually 1, and the x is scaled appropriately
 - For example 2.2:1
 - That probability is $1/(1+2.2) = 1/3.2 = 0.313$
- 1:1 odds means that you will lose as many times as you win

41
