Basics of Counting

22C:19, Chapter 6.5, 6.7
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1

A note on combinations

- An alternative (and more common) way to denote an *r*-combination:

$$C(n,r) = \binom{n}{r}$$

- I'll use C(*n*,*r*) whenever possible, as it is easier to write in PowerPoint

2

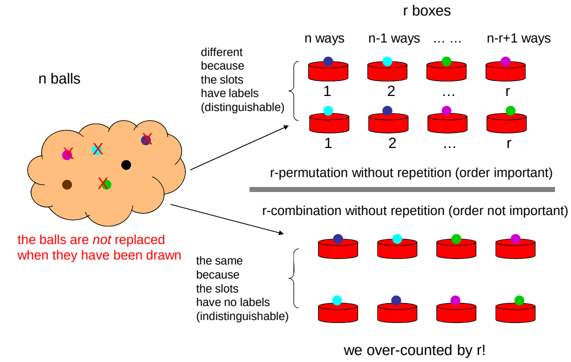
Recap

r-permutations: The number of ways in which we can draw r balls from a collection of n different balls, where the order is important: P(n,r) = n! / (n-r)!

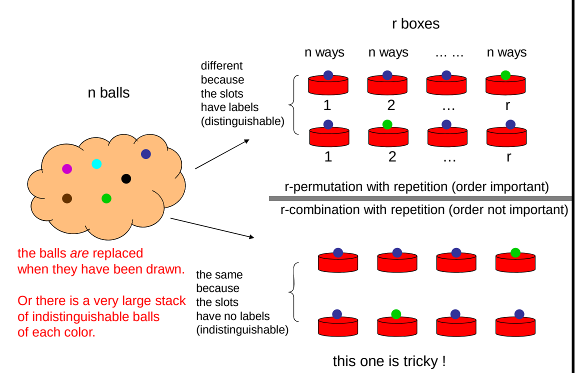
r-combinations: The number of ways in which we can draw r balls from a collection of n different balls, where we do not care about the ordering: C(n,r) = n! / r! (n-r) !

Today: we study counting problems where *repetitions* are allowed, i.e. it is possible that the same ball is drawn multiple times.

Combinations without repetition



Combinations with repetition



Combinations with repetition

Example: We want to draw 2 pieces of fruit from a bowl that contains 2 apples, 2 pears, and 2 oranges. In how many ways can we do this such that:

1) The fruits are numbered and the order matters.
2) The fruits are numbered and the order does not matter.
3) The fruits are indistinguishable but the order matters.
4) The fruits are indistinguishable and the order does not matter.

→ 1) 6 * 5 = 30 P(6,2)
→ 2) 6 * 5 / 2 = 15 C(6,2)

→ 3) Now there are 3 *kinds of* fruit that we draw with replacement (since there are enough of each kind to be able to pick any fruit at any draw). This is true because drawing apple 1 is no different than drawing apple 2. It's like there are copies of the same apple present.
Thus: $3^2 = 9$
(a a), (a p), (a o), (p a), (p p), (p o), (o a), (o p), (o o).

→ 4) Since the order doesn't matter (a p) = (p a), (a o)=(o a), (p o)=(o p).
we over-counted 3 pieces: 9-3 = 6.

Combinations with repetition

How to count the latter (r-combination with repetition)?

One strategy could be to start from drawing where the order matters and try to count the number of ways we over-counted (last example).

However, there is a much smarter way!

Three kinds of fruits are separated by two dividers and two chosen fruits are held by two boxes: We choose two items to fill the boxes.

apples | pears | oranges

X X

(a a) = XX | |, (p p) = | XX |,
(a p) = X | X |, (p o) = | X | X,
(a o) = X | | X, (o o) = | | XX

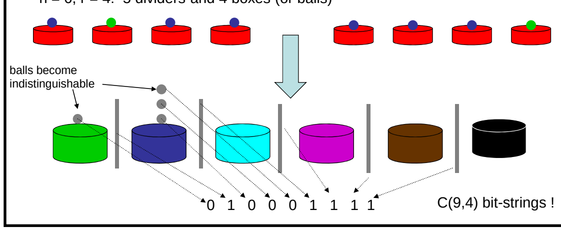
The total number of "X" and "|" is 4.
The number of ways choosing 2 positions out of 4 is C(4,2) = 6.
We may also use "0" for "X" and "1" for "|".

Combinations with repetition

How to count the latter (r-combination with repetition)?

The previous example can be generalized: Choose r balls out of a bag of balls of n different colors needs (n-1) dividers ("|") and r boxes ("X"). The total number of "|" and "X" is (n+r-1). The number of ways choosing r positions out of (n+r-1) positions is C(n+r-1,r) = (n+r-1)! /(r! (n-1)!)

n = 6, r = 4: 5 dividers and 4 boxes (or balls)



More examples

1) How many ways are there to select five bills from a cash box containing many $1, $2, $5, $10, $20, $50 and $100 bills, such that the bills of the same value are indistinguishable and the order in which they are selected is unimportant.

→ This is like drawing colored balls <u>with replacement</u>. The colors correspond to the values. Since the order doesn't matter we have: C(7+5-1,5)=462

2) A cookie shop has 4 <u>kinds</u> of cookies and we want to pick 6. We don't care about the order and cookies from one kind are indistinguishable.

→ Again, drawing colored balls <u>with replacement</u>: colors are kind of cookies.
C(6+4-1,6)=84.

Combinations with repetition

n is number of distinct classes of objects in the original bag!

- r-permutation without repetition - order matters (r distinguishable slots) - without replacement (n distinguishable objects) $n! / (n-r)!$	- r-combination without repetition - order does not matter (r indistinguishable slots) - without replacement (n distinguishable objects) $n! / r! (n-r)!$
- r-permutation with repetition - order matters (r distinguishable slots) - with replacement (n distinct classes of indistinguishable objects) n^r	- r-combination with repetition - order does not matter (r indistinguishable slots) - with replacement (n distinct classes of indistinguishable objects) $(n+r-1)! / r! (n-1)!$

Another example

How many different non-negative integer solutions for the variables x1, x2, x3, x4 with x1 + x2 + x3 + x4 = 10?

A solution like x1 = 1, x2 = 0, x3 = 4, x4 = 5 divides 10 into four parts: (1, 0, 4, 5) or “X | | XXXX | XXXXX”.

We need three dividers (“|”) to divide 10 boxes (“X”) into four parts. The number of ways of choosing three slots out of 10+3 slots is
C(10+4-1, 3) = C(13, 3) = 286.

11

Another example

How many different integer triples (i, j, k) where 1<= i <= j <= k <= n?

Let n = 5. A triple like (3, 3, 5) may be viewed as

| | XX | | X

which have three boxes (“X”) holding objects from five classes (1 thru 5) and we need four dividers to separate these five classes.

In general, we need n-1 dividers (“|”) to divide three boxes (“X”) into n parts.
The number of ways of choosing three slots out of (n-1)+3 slots is C(n+2, 3) = (n+2)(n+1)n/3.

12

Another example

How many times will the code B in the following program be executed?

```
For k :=  1 to n
   For j := 1 to k
      For  i := 1 to j
           B(i,j,k)
       end for
   end for
end for
```

Answer: C(n+2, 3) = (n+2)(n+1)n/3.

13

Book-Shelve Problems

In how many ways can you put n different books on k different shelves? (shelves can hold all books).

Solution: Place books one by one.
First book: on k shelves.
Second book: to the left or right of existing book, or on empty shelve (2+k-1=k+1).
Third book: Two cases:
- (a) The first two books on one shelve: 3 ways on the same shelf and k-1 ways on an empty shelve. 3+(k-1) = k+2.
- (b) The first two books not one shelve: 2 ways with book1 and two ways with book2, (k-2) ways on an empty shelve. 2+2+(k-2) = k+2.

Fourth book: k+3 ways.
…
Total: k(k+1)(k+2) ... (k+n-1) = (n+k-1)! / (k-1)!

Book-Shelve Problems

In how many ways can you put n different books on k different shelves? (shelves can hold all books).

Solution 2: There are n! ways to put them into a sequence. For each sequence, we need to cut the sequence into k subsequences using k-1 dividers.

→ how many bit-strings are there with k-1 “|” (dividers) and n “X” (books): C(n+k-1,k-1).

Total: C(n+k-1,k-1) n! = (n+k-1)!/(k-1)!

Subsets

- If all the elements of a set S are also elements of a set T, then S is a subset of T
  - For example, if S = {2, 4, 6} and T = {1, 2, 3, 4, 5, 6, 7}, then S is a subset of T
  - This is specified by $S \subseteq T$
    - Or by $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$
- If S is not a subset of T, it is written as such: $S \nsubseteq T$
  - For example, $\{1, 2, 8\} \nsubseteq \{1, 2, 3, 4, 5, 6, 7\}$

16

Subsets

- Any set is a subset of itself!
  - Given set S = {2, 4, 6}, since all the elements of S are elements of S, S is a subset of itself
  - This is kind of like saying 5 is less than or equal to 5
  - Thus, for any set S, $S \subseteq S$
- The empty set is a subset of *all* sets (including itself!)
  - Recall that all sets are subsets of themselves
- *All* sets are subsets of the universal set.

17

Proper Subsets

- If S is a subset of T, and S is not equal to T, then S is a proper subset of T
  - Let T = {0, 1, 2, 3, 4, 5}
  - If S = {1, 2, 3}, S is not equal to T, and S is a subset of T
  - A proper subset is written as $S \subset T$
  - Let R = {0, 1, 2, 3, 4, 5}. R is equal to T, and thus is a subset (but not a proper subset) or T
    - Can be written as: $R \subseteq T$ and $R \not\subset T$ (or just R = T)
  - Let Q = {4, 5, 6}. Q is neither a subset or T nor a proper subset of T

18

Set cardinality

- The cardinality of a set is the number of elements in a set
  - Written as |A|
- Examples
  - Let R = {1, 2, 3, 4, 5}. Then |R| = 5
  - |∅| = 0
  - Let S = {∅, {a}, {b}, {a, b}}. Then |S| = 4
- This is the same notation used for vector length in geometry
- A set with one element is sometimes called a singleton set

19

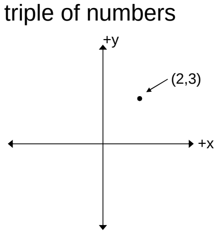
Power sets

- Given the set S = {0, 1}. What are all the possible subsets of S?
  - They are: ∅ (as it is a subset of all sets), {0}, {1}, and {0, 1}
  - The power set of S (written as P(S)) is the set of all the subsets of S
  - P(S) = { ∅, {0}, {1}, {0,1} }
- Theorem: If |S| = n, then |P(S)| = 2^n

20

Tuples

- In 2-dimensional space, it is a (*x*, *y*) pair of numbers to specify a location
- In 3-dimensional (1,2,3) is not the same as (3,2,1) – space, it is a (*x*, *y*, *z*) triple of numbers
- In *n*-dimensional space, it is a *n*-tuple of numbers
  - Two-dimensional space uses pairs, or 2-tuples
  - Three-dimensional space uses triples, or 3-tuples
- Note that these tuples are **ordered**, unlike sets
  - the *x* value has to come first



21

Cartesian products

- A Cartesian product is a set of all ordered 2-tuples where each "part" is from a given set
  - Denoted by A x B, and uses parenthesis (not curly brackets)
  - For example, 2-D Cartesian coordinates are the set of all ordered pairs **Z** x **Z**
    - Recall **Z** is the set of all integers
    - This is all the possible coordinates in 2-D space
  - Example: Given A = { a, b } and B = { 0, 1 }, what is their Cartiesian product?
    - C = A x B = { (a,0), (a,1), (b,0), (b,1) }

22

Cartesian products

- Note that Cartesian products have only 2 parts in these examples (later examples have more parts)
- Formal definition of a Cartesian product:
  - A x B = { (*a*,*b*) | *a* ∈ A and *b* ∈ B }
- Theorem: |A x B| = |A||B|.

23

Cartesian products

- All the possible grades in this class will be a Cartesian product of the set S of all the students in this class and the set G of all possible grades
  - Let S = { Alice, Bob, Chris } and G = { A, B, C }
  - D = { (Alice, A), (Alice, B), (Alice, C), (Bob, A), (Bob, B), (Bob, C), (Chris, A), (Chris, B), (Chris, C) }
  - The final grades will be a subset of this: { (Alice, C), (Bob, B), (Chris, A) }
    - Such a subset of a Cartesian product is called a relation (more on this later in the course)

24

Cartesian products

- There can be Cartesian products on more than two sets
- A 3-D coordinate is an element from the Cartesian product of **Z** x **Z** x **Z**

25

The Binomial Theorem

- Theorem: Given any numbers *a* and *b* and any nonnegative integer *n*,

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

The Binomial Theorem

- Proof: Use induction on *n*.
- Base case: Let *n* = 0. Then
  - $(a + b)^0 = 1$ and

$$\sum_{i=0}^{0} \binom{0}{i} a^{0-i} b^i = \binom{0}{0} a^{0-0} b^0 = 1.$$

  - Therefore, the statement is true when *n* = 0.

Proof, continued

- Inductive step
  - Suppose the statement is true when $n = k$ for some $k \geq 0$.
  - Then

$$(a+b)^{k+1} = (a+b)(a+b)^k$$
$$= (a+b)\sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i$$
$$= \sum_{i=0}^{k}\binom{k}{i}a^{k-i+1}b^i + \sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^{i+1}$$

Proof, continued

$$= a^{k+1} + \sum_{i=1}^{k}\binom{k}{i}a^{k-i+1}b^i + \sum_{i=0}^{k-1}\binom{k}{i}a^{k-i}b^{i+1} + b^{k+1}$$
$$= a^{k+1} + \sum_{i=1}^{k}\binom{k}{i}a^{k-i+1}b^i + \sum_{i=1}^{k}\binom{k}{i-1}a^{k-i+1}b^i + b^{k+1}$$
$$= a^{k+1} + \sum_{i=1}^{k}\left(\binom{k}{i}+\binom{k}{i-1}\right)a^{k-i+1}b^i + b^{k+1}$$

Proof, continued

$$= a^{k+1} + \sum_{i=1}^{k}\binom{k+1}{i}a^{k-i+1}b^i + b^{k+1}$$
$$= \sum_{i=0}^{k+1}\binom{k+1}{i}a^{k-i+1}b^i.$$

  - Therefore, the statement is true when $n = k + 1$.
- Thus, the statement is true for all $n \geq 0$.

Example: Binomial Theorem

- Expand $(a + b)^8$.
  - $C(8, 0) = C(8, 8) = 1$.
  - $C(8, 1) = C(8, 7) = 8$.
  - $C(8, 2) = C(8, 6) = 28$.
  - $C(8, 3) = C(8, 5) = 56$.
  - $C(8, 4) = 70$.

Example: Binomial Theorem

- Therefore,

$$(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8.$$

Example: Calculating 1.01^8

- Compute 1.01^8 on a calculator.
- What do you see?

Example: Calculating 1.01^8

- Compute 1.01^8 on a calculator.
- What do you see?
- $1.01^8 = 1.0828567056280801$.

Example: Calculating 1.01^6

- $1.01^8 = (1 + 0.01)^8$

  $= 1 + 8(0.01) + 28(0.01)^2 + 56(0.01)^3$

  $+ 70(0.01)^4 + 56(0.01)^5 +$

  $+ 28(0.01)^6 + 8(0.01)^7 + (0.01)^8$

  $= 1 + .08 + .0028 + .000056 + .00000070$

  $+ .0000000056 + .000000000028 +$

  $+ .00000000000008$

  $+ .0000000000000001$

  $= 1.0828567056280801.$

Example: Approximating $(1+x)^n$

- Theorem: For small values of x,

$$(1+x)^n \approx 1 + nx.$$

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2.$$

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3.$$

and so on.

Example

- For example,

$$(1 + x)^8 \approx 1 + 8x + 28x^2$$

  when x is small.
- Compute the value of $(1 + x)^8$ and the approximation when $x = .03$.
- Do it again for $x = -.03$.

Expanding Trinomials

- Expand $(a + b + c)^3$.

Expanding Trinomials

- Expand $(a + b + c)^3$.
- $(a + b + c)^3 = ((a + b) + c)^3$

$$\begin{aligned} &= (a + b)^3 + 3(a + b)^2c \\ &\quad + 3(a + b)c^2 + c^3, \\ &= (a^3 + 3a^2b + 3ab^2 + b^3) \\ &\quad + 3(a^2 + 2ab + b^2)c \\ &\quad + 3(a + b)c^2 \\ &\quad + c^3. \end{aligned}$$

Expanding Trinomials

$= a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc$
$+ 3b^2c + 3ac^2 + 3bc^2 + c^3.$

- What is the pattern?

Expanding Trinomials

- $(a + b + c)^3 = (a^3 + b^3 + c^3)$
  $+ 3(a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2)$
  $+ 6abc.$

The Multinomial Theorem

- Theorem: In the expansion of
  $$(a_1 + \ldots + a_k)^n,$$
  the coefficient of $a_1^{n_1}a_2^{n_2}\ldots a_k^{n_k}$ is
  $$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Example: The Multinomial Theorem

- Expand $(a + b + c + d)^3$.
- The terms are
  - a^3, b^3, c^3, d^3, with coefficient 3!/3! = 1.
  - a^2b, a^2c, a^2d, ab^2, b^2c, b^2d, ac^2, bc^2, c^2d, ad^2, bd^2, cd^2, with coefficient 3!/(1!2!) = 3.
  - abc, abd, acd, bcd, with coefficient 3!/(1!1!1!) = 6.

Example: The Multinomial Theorem

- Therefore,

$$(a + b + c + d)^3 = a^3 + b^3 + c^3 + d^3 + 3a^2b + 3a^2c + 3a^2d + 3ab^2 + 3b^2c + 3b^2d + 3ac^2 + 3bc^2 + 3c^2d + 3ad^2 + 3bd^2 + 3cd^2 + 6abc + 6abd + 6acd + 6bcd.$$

Example: The Multinomial Theorem

- Find $(a + 2b + 1)^4$.

Another Problem

- If we expand the expression
  $(a + 2b + 3c)^4$,
  what will be the sum of the coefficients?