

Set Operations

22c:19 Chapter 5
Hantao Zhang

Universe of Reference

When talking about a set, a **universe of reference (universal set)** needs to be specified. Even though a set is defined by the elements which it contains, those elements cannot be arbitrary. If arbitrary elements are allowed paradoxes can result arising from self reference.

L5 2

Russel's Paradox

If we allow arbitrary elements, we should also allow sets to be elements, and sets of sets, and so on. So it would be perfectly reasonable to consider the following set:

S = the set containing all sets which do not contain themselves

CLAIM: This set cannot exist.

Proof: If it existed, it would either contain itself, or not. Let's consider both cases:

$S \in S$: a contradiction to the definition of S .

$S \notin S$: By the definition of S , we must have $S \in S$.

L5 3

Russel's Paradox

It is of paramount importance that when a set is specified by stating certain conditions, that it ought to exist as a set. The set could be empty, but that's fine, as long as it actually exists.

Example: The set of all pigs which can fly. The condition is not achievable so that this set is the empty set \emptyset . It is still a set, however.

To guarantee set existentially, a universal set U should always be fixed.

L5

4

Set Builder Notation

Sets can be defined using the curly brace notation "{ ... }" or descriptively "the set of all natural numbers". The set builder notation allows for concise definition of new sets.

For example

- { $x \mid x$ is an even integer }
- { $2x \mid x$ is an integer }

are equivalent ways of specifying the set of all even integers.

L5

5

Set Builder Notation

In general, one specifies a set by writing

$$\{ f(x) \mid P(x) \}$$

Where $f(x)$ is a function of x and $P(x)$ is a propositional function of x . The notation is read as

"the set of all elements $f(x)$ such that $P(x)$ holds"

- Stuff between "{ " and "| "
- specifies how elements look
- Stuff between the "| " and " } "
- gives properties elements satisfy
- Pipe symbol "| " is
- short-hand for "such that".

L5

6

**Set Builder Notation.
Shortcuts.**

- To specify a subset of a pre-defined set, we use, for example
 $\{x \in \mathbf{N} \mid \exists y (x = 2y)\}$
 which is a shorthand for
 $\{x \mid x \in \mathbf{N} \text{ and } \exists y (x = 2y)\}$
 Both define the set of all even natural numbers (assuming universe of reference \mathbf{Z}).
- When universe of reference is understood, don't need to specify propositional function.

For example: $\{x^3 \mid x \in \mathbf{N}\}$ or simply $\{x^3\}$ specifies the set of perfect cubes
 $\{0, 1, 8, 27, 64, 125, \dots\}$
 assuming U is the set of natural numbers.

L5 7

**Set Builder Notation.
Examples.**

Q1: $U = \mathbf{N}$. $\{x \mid \forall y (y \geq x)\} = ?$
 Q2: $U = \mathbf{Z}$. $\{x \mid \forall y (y \geq x)\} = ?$
 Q3: $U = \mathbf{Z}$. $\{x \mid \exists y (y \in \mathbf{R} \wedge y^2 = x)\} = ?$
 Q4: $U = \mathbf{Z}$. $\{x \mid \exists y (y \in \mathbf{R} \wedge y^3 = x)\} = ?$
 Q5: $U = \mathbf{R}$. $\{|x| \mid x \in \mathbf{Z}\} = ?$
 Q6: $U = \mathbf{R}$. $\{|x|\} = ?$

L5 8

**Set Builder Notation.
Examples.**

A1: $U = \mathbf{N}$. $\{x \mid \forall y (y \geq x)\} = \{0\}$
 A2: $U = \mathbf{Z}$. $\{x \mid \forall y (y \geq x)\} = \{\}$
 A3: $U = \mathbf{Z}$. $\{x \mid \exists y (y \in \mathbf{R} \wedge y^2 = x)\}$
 $= \{0, 1, 2, 3, 4, \dots\} = \mathbf{N}$
 A4: $U = \mathbf{Z}$. $\{x \mid \exists y (y \in \mathbf{R} \wedge y^3 = x)\} = \mathbf{Z}$
 A5: $U = \mathbf{R}$. $\{|x| \mid x \in \mathbf{Z}\} = \mathbf{N}$
 A6: $U = \mathbf{R}$. $\{|x|\} = \text{non-negative reals.}$

L5 9

Set Theoretic Operations

Set theoretic operations allow us to build new sets out of old, just as the logical connectives allowed us to create compound propositions from simpler propositions. Given sets A and B , the set theoretic operators are:

- Union (\cup)
- Intersection (\cap)
- Difference ($-$)
- Complement ("—")
- Symmetric Difference (\oplus)

give us new sets $A \cup B, A \cap B, A - B, A \oplus B$, and \bar{A} .

L5

10

Venn Diagrams

Venn diagrams are useful in representing sets and set operations. Various sets are represented by circles inside a big rectangle representing the universe of reference.

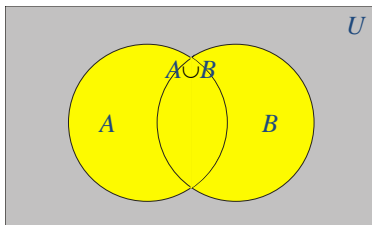
L5

11

Union

Elements in at least one of the two sets:

$$A \cup B = \{ x / x \in A \vee x \in B \}$$

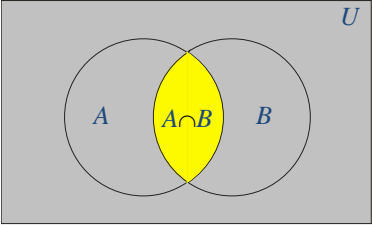


L5

12

Intersection

Elements in exactly one of the two sets:
 $A \cap B = \{ x / x \in A \wedge x \in B \}$



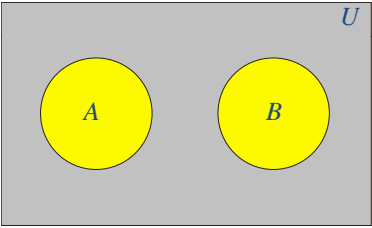
U

A *A ∩ B* *B*

L5 13

Disjoint Sets

DEF: If *A* and *B* have no common elements, they are said to be **disjoint**, i.e. $A \cap B = \emptyset$.



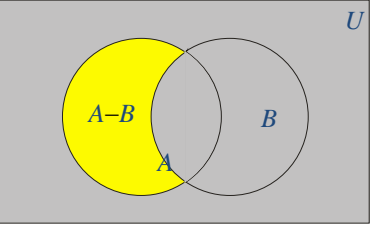
U

A *B*

L5 14

Set Difference

Elements in first set but not second:
 $A - B = \{ x / x \in A \wedge x \notin B \}$



U

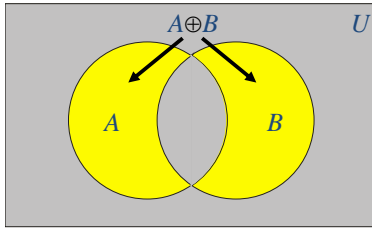
A - B *A* *B*

L5 15

Symmetric Difference

Elements in exactly one of the two sets:

$$A \oplus B = \{ x / x \in A \oplus x \in B \}$$



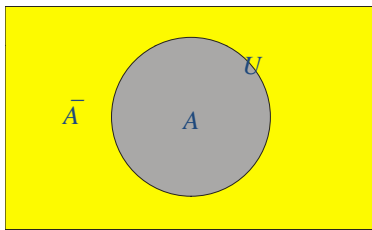
L5

16

Complement

Elements not in the set (unary operator):

$$\bar{A} = \{ x / x \notin A \}$$



L5

17

Set Identities

- Identity laws
- Domination laws
- Idempotent laws
- Double complementation
- Commutativity
- Associativity
- Distributivity
- DeMorgan
 - disjunction " \vee " becomes union " \cup "
 - conjunction " \wedge " becomes intersection " \cap "
 - negation " \neg " becomes complementation " $\bar{}$ "
 - false "F" becomes the empty set \emptyset
 - true "T" becomes the universe of reference U

L5

18

Set Identities

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)
 $(A \cup B) \cup C = A \cup (B \cup C)$

L5

19

Set Identities

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)
 $(A \cup B) \cup C = A \cup (B \cup C)$
Proof: $(A \cup B) \cup C = \{x \mid x \in A \cup B \vee x \in C\}$ (by def.)

L5

20

Set Identities

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)
 $(A \cup B) \cup C = A \cup (B \cup C)$
Proof: $(A \cup B) \cup C = \{x \mid x \in A \cup B \vee x \in C\}$ (by def.)
 $= \{x \mid (x \in A \vee x \in B) \vee x \in C\}$ (by def.)

L5

21

Set Identities

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof: $(A \cup B) \cup C = \{x \mid x \in A \cup B \vee x \in C\}$ (by def.)
 $= \{x \mid (x \in A \vee x \in B) \vee x \in C\}$ (by def.)
 $= \{x \mid x \in A \vee (x \in B \vee x \in C)\}$ (logical assoc.)

L5 22

Set Identities

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof: $(A \cup B) \cup C = \{x \mid x \in A \cup B \vee x \in C\}$ (by def.)
 $= \{x \mid (x \in A \vee x \in B) \vee x \in C\}$ (by def.)
 $= \{x \mid x \in A \vee (x \in B \vee x \in C)\}$ (logical assoc.)
 $= \{x \mid x \in A \vee x \in B \cup C\}$ (by def.)

L5 23

Set Identities

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof: $(A \cup B) \cup C = \{x \mid x \in A \cup B \vee x \in C\}$ (by def.)
 $= \{x \mid (x \in A \vee x \in B) \vee x \in C\}$ (by def.)
 $= \{x \mid x \in A \vee (x \in B \vee x \in C)\}$ (logical assoc.)
 $= \{x \mid x \in A \vee x \in B \cup C\}$ (by def.)
 $= A \cup (B \cup C)$ (by def.)

□
 Other identities are derived similarly.

L5 24

Set Identities via Venn

It's often simpler to understand an identity by drawing a Venn Diagram.

For example DeMorgan's first law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

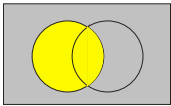
can be visualized as follows.

L5

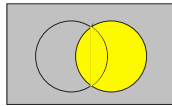
25

Visual DeMorgan

A:



B:

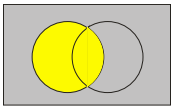


L5

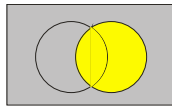
26

Visual DeMorgan

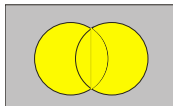
A:



B:

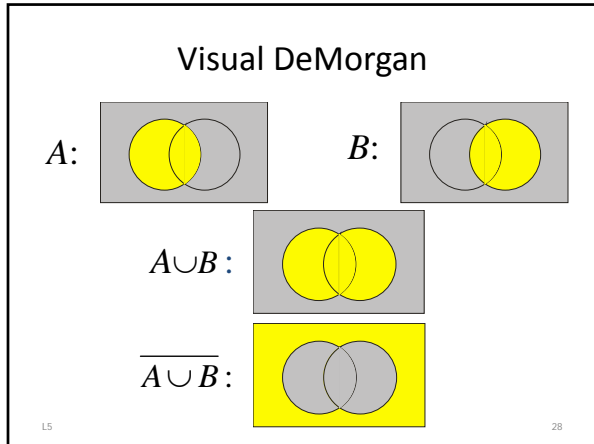


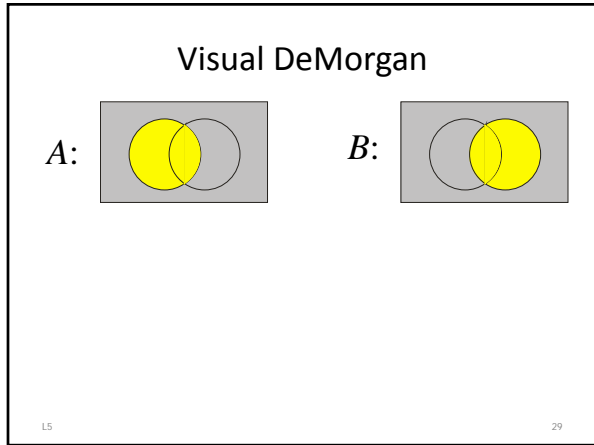
$A \cup B$:

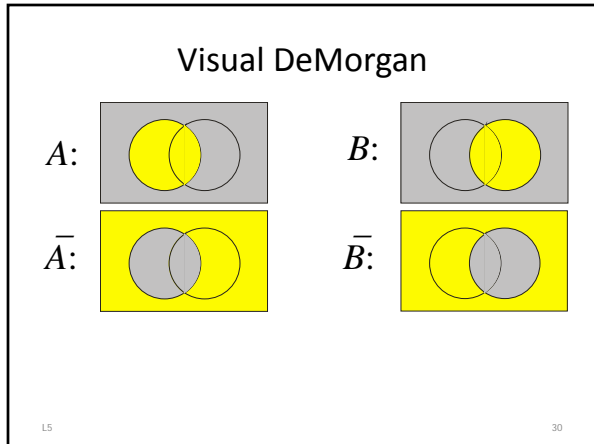


L5

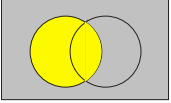
27

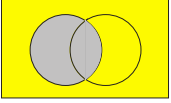


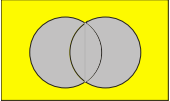


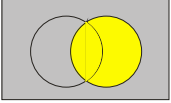


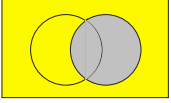
Visual DeMorgan

A : 

\bar{A} : 

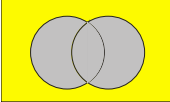
$\bar{A} \cap \bar{B}$: 

B : 

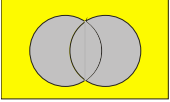
\bar{B} : 

L5 31

Visual DeMorgan

$\overline{A \cup B} =$


=

$\bar{A} \cap \bar{B} =$


L5 32

Sets as Bit-Strings

If we order the elements of our universe, we can represent sets by bit-strings. For example, consider the universe

$U = \{\text{ant, beetle, cicada, dragonfly}\}$

Order the elements alphabetically. Subsets of U are represented by bit-strings of length 4. Each bit in turn, tells us whether the corresponding element is contained in the set. EG: {ant, dragonfly} is represented by the bit-string 1001.

Q: What set is represented by 0111 ?

L5 33

Sets as Bit-Strings

A: 0111 represents
{beetle, cicada, dragonfly}

Conveniently, under this representation the various set theoretic operations become the logical bit-string operators that we saw before. For example, the symmetric difference of {beetle} with {ant, beetle, dragonfly} is represented by:

$$\begin{array}{r} 0100 \\ \oplus \quad \underline{1101} \\ 1001 = \{\text{ant, dragonfly}\} \end{array}$$

L5

34
