Mathematical Induction

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Chapter 4
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How do you program a robot to climb infinite stairs?

- First, you let the robot get to the base platform of the staircase
- Then repeat:
  - From the current position, let the robot to move one step up

Let's use that as a proof method

- First, show P(x) is true for *x*=0
  - This is the base of the stairs
- Then, show that if it's true for some value *n*, then it is true for *n*+1
  - Show: P(*n*) → P(*n*+1)
  - This is climbing the stairs
  - Let *n*=0. Since it's true for P(0) (base case), it's true for *n*=1
  - Let *n*=1. Since it's true for P(1) (previous bullet), it's true for *n*=2
  - Let *n*=2. Since it's true for P(2) (previous bullet), it's true for *n*=3
  - Let *n*=3 …
  - And onwards to infinity
- Thus, we have shown it to be true for *all* non-negative numbers

What is induction?

- A method of proof
- It does not generate answers: it only can prove them
- Three parts:
  - Base case(s): show it is true for one element
    - (get to the stair's base platform)
  - Inductive hypothesis: assume it is true for any given element
    - (assume you are on a stair)
    - **Must be clearly labeled!!!**
  - Show that if it true for the next higher element
    - (show you can move to the next stair)

Induction example

- Show that the sum of the first *n* odd integers is n^2
  - Example: If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, show:

$$\forall n \, \mathrm{P}(n) \qquad \text{where} \quad \mathrm{P}(n) = \sum_{i=1}^{n} 2i - 1 == n^2$$

- Base case: Show that P(1) is true

$$\mathrm{P}(1) = \sum_{i=1}^{1} 2(i) - 1 == 1^2$$

$$= 1 == 1$$

Induction example, continued

- Inductive hypothesis: assume true for *k*
  - Thus, we assume that P(*k*) is true, or that

$$\sum_{i=1}^{\boxed{k}} 2i - 1 == \boxed{k}^2$$

  - Note: we don't yet know if this is true or not!
- Inductive step: show true for *k*+1
  - We want to show that:

$$\sum_{i=1}^{\boxed{k+1}} 2i - 1 == \boxed{(k+1)}^2$$

Induction example, continued

- Recall the inductive hypothesis: $\sum_{i=1}^{k} 2i-1 == k^2$
- Proof of inductive step:

$$\sum_{i=1}^{k+1} 2i-1 == (k+1)^2$$

$$2(k+1)-1+\boxed{\sum_{i=1}^{k} 2i-1} == k^2+2k+1$$

$$2(k+1)-1+\boxed{k^2} == k^2+2k+1$$

$$k^2+2k+1 == k^2+2k+1$$

What did we show

- Base case: P(1)
- If P(*k*) was true, then P(*k*+1) is true
  - i.e., P(*k*) → P(*k*+1)

- We know it's true for P(1)
- Because of P(*k*) → P(*k*+1), if it's true for P(1), then it's true for P(2)
- Because of P(*k*) → P(*k*+1), if it's true for P(2), then it's true for P(3)
- Because of P(*k*) → P(*k*+1), if it's true for P(3), then it's true for P(4)
- Because of P(*k*) → P(*k*+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it is true for all possible values of *n*

- In other words, we showed that:

$$\left[\mathrm{P}(1) \wedge \forall k\left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)\right] \rightarrow \forall n\, \mathrm{P}(n)$$

The idea behind inductive proofs

- Show the base case
- Show the inductive hypothesis
- Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
- Show the inductive step

Second induction example

- Show the sum of the first *n* positive even integers is $n^2 + n$
  - Rephrased:

$$\forall n\ \mathrm{P}(n) \qquad \text{where } \mathrm{P}(n) = \sum_{i=1}^{n} 2i == n^2 + n$$

- The three parts:
  - Base case
  - Inductive hypothesis
  - Inductive step

Second induction example, continued

- Base case: Show P(1):

$$\mathrm{P}(1) = \sum_{i=1}^{1} 2(i) == 1^2 + 1$$
$$= 2 == 2$$

- Inductive hypothesis: Assume

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

- Inductive step: Show

$$\mathrm{P}(k+1) = \sum_{i=1}^{k+1} 2i == (k+1)^2 + (k+1)$$

Second induction example, continued

- Recall our inductive hypothesis: $\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$
$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$
$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$
$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Notes on proofs by induction

- We manipulate the *k*+1 case to make part of it look like the *k* case
- We then replace that part with the other side of the *k* case

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Third induction example

- Show $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- Base case: *n* = 1

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$$

$$1^2 = \frac{6}{6}$$

$$1 = 1$$

- Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction example

- Inductive step: show $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$$

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Show: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+\boxed{2})}{6}$
- Base case: $n = 1$:

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+2)}{6}$$
$$1^2 = \frac{7}{6}$$
$$1 \neq \frac{7}{6}$$

- But let's continue anyway…
- Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

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Third induction again: what if your inductive hypothesis was wrong?

- Inductive step: show $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$
$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+4)}{6}$$
$$(k+1)^2 + \frac{k(k+1)(2k+2)}{6} = \frac{(k+1)(k+2)(2k+4)}{6}$$
$$6(k+1)^2 + k(k+1)(2k+2) = (k+1)(k+2)(2k+4)$$
$$2k^3 + 10k^2 + 14k + 6 \neq 2k^3 + 10k^2 + 16k + 8$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Fourth induction example

- S that $n! < n^n$ for all $n > 1$
- Base case: $n = 2$

  $2! < 2^2$

  $2 < 4$
- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

$$(k+1)! = (k+1)\boxed{k!} < (k+1)\boxed{k^k} < (k+1)(k+1)^k = (k+1)^{k+1}$$

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Strong induction

- Weak mathematical induction proves P(0) is true and assumes P(*k*) is true, and uses that (and only that!) to show P(*k*+1) is true
- Strong mathematical induction proves P(0), P(1), …, P(b) are true, and assumes P(*k*-*b*), P(*k*-*b*+1), …, P(*k*) are all true, and uses that to show that P(*k*+1) is true.

$$\left[\mathrm{P}(k-b) \wedge \mathrm{P}(k-b+1) \wedge \mathrm{P}(k-b+2) \wedge \ldots \wedge \mathrm{P}(k)\right] \rightarrow \mathrm{P}(k+1)$$

Strong induction vs. non-strong induction

- Determine which amounts of postage can be written with 5 and 6 cent stamps
  - Prove using both versions of induction
- Answer: any postage ≥ 20 plus 5, 6, 10, 11, 12, 15, 16, 17, 18.

Answer via mathematical induction

- Show base case: P(20):
  - 20 = 5 + 5 + 5 + 5
- Inductive hypothesis: Assume P(*k*) is true
- Inductive step: Show that P(*k*+1) is true
  - If P(*k*) uses a 5 cent stamp, replace that stamp with a 6 cent stamp
  - If P(*k*) does not use a 5 cent stamp, it must use only 6 cent stamps
    - Since *k* > 18, there must be four 6 cent stamps
    - Replace these with five 5 cent stamps to obtain *k*+1

Answer via strong induction

- Show base cases: P(20), P(21), P(22), P(23), and P(24)
  - 20 = 5 + 5 + 5 + 5
  - 21 = 5 + 5 + 5 + 6
  - 22 = 5 + 5 + 6 + 6
  - 23 = 5 + 6 + 6 + 6
  - 24 = 6 + 6 + 6 + 6
- Inductive hypothesis: Assume P(*k*-4), P(*k*-3), …, P(*k*) are all true
- Inductive step: Show that P(*k*+1) is true, where *k*+1>24
  - We will obtain P(*k*+1) by adding a 5 cent stamp to P(*k*+1-5)
  - Since we know P(*k*+1-5) = P(*k*-4) is true, where *k*-4>19, our proof is complete

Strong induction vs. non-strong induction, take 2

- Show that every postage amount 12 cents or more can be formed using only 4 and 5 cent stamps
- Similar to the previous example

Answer via mathematical induction

- Show base case: P(12):
  - 12 = 4 + 4 + 4
- Inductive hypothesis: Assume P(*k*) is true
- Inductive step: Show that P(*k*+1) is true
  - If P(*k*) uses a 4 cent stamp, replace that stamp with a 5 cent stamp to obtain *P*(*k*+1)
  - If P(*k*) does not use a 4 cent stamp, it must use only 5 cent stamps
    - Since *k* > 10, there must be at least three 5 cent stamps
    - Replace these with four 4 cent stamps to obtain *k*+1
- Note that only *P*(*k*) was assumed to be true

Answer via strong induction

- Show base cases: P(12), P(13), P(14), and P(15)
  - 12 = 4 + 4 + 4
  - 13 = 4 + 4 + 5
  - 14 = 4 + 5 + 5
  - 15 = 5 + 5 + 5
- Inductive hypothesis: Assume P(*k*-3), P(*k*-2), P(*k*-1), P(*k*) are all true
  - For $k \geq 15$
- Inductive step: Show that P(*k*+1) is true
  - We will obtain P(*k*+1) by adding a 4 cent stamp to P(*k*+1-4)
  - Since we know P(*k*+1-4) = P(*k*-3) is true, our proof is complete
- Note that P(*k*-3), P(*k*-2), P(*k*-1), P(*k*) were all assumed to be true

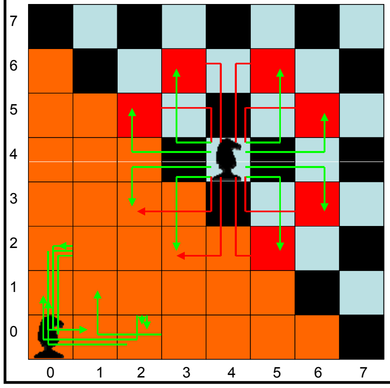
Strong induction

- Weak mathematical induction proves P(0) is true and assumes P(*k*) is true, and uses that (and only that!) to show P(*k*+1) is true

- Strong mathematical induction proves P(0), P(1), …, P(b) are true, and assumes P(*k*-*b*), P(*k*-*b*+1), …, P(*k*) are all true, and uses that to show that P(*k*+1) is true.

$$[\mathrm{P}(k-b) \wedge \mathrm{P}(k-b+1) \wedge \mathrm{P}(k-b+2) \wedge \ldots \wedge \mathrm{P}(k)] \rightarrow \mathrm{P}(k+1)$$

Chess and induction



Can the knight reach any square in a finite number of moves?

Show that the knight can reach any square (*i*, *j*) for which *i*+*j*=*k* where *k* > 1.

Base cases: *k* = 0,1,2

Inductive hypothesis: assume the knight can reach any square (*i*, *j*) for which *i*+*j*=*k* where *k* > 1.

Inductive step: show the knight can reach any square (*i*, *j*) for which *i*+*j*=*k*+1 where *k* > 1.

Chess and induction

- Inductive step: show the knight can reach any square (i, j) for which $i+j=k+1$ where $k > 1$.
  - Note that $k+1 \geq 3$, and one of i or j is ≥ 2
  - If $i \geq 2$, the knight could have moved from (i-2, j+1)
    - Since $i+j = k+1$, $i-2 + j+1 = k$, which is assumed true
  - If $j \geq 2$, the knight could have moved from (i+1, j-2)
    - Since $i+j = k+1$, $i+1 + j-2 = k$, which is assumed true

Strong induction (compact form)

- Strong mathematical induction proves P(0), P(1), …, P(b) are true, and assumes P(k-b), P(k-b+1), …, P(k) are all true, and uses that to show that P(k+1) is true.

$$[\mathrm{P}(k-b) \wedge \mathrm{P}(k-b+1) \wedge \mathrm{P}(k-b+2) \wedge \ldots \wedge \mathrm{P}(k)] \rightarrow \mathrm{P}(k+1)$$

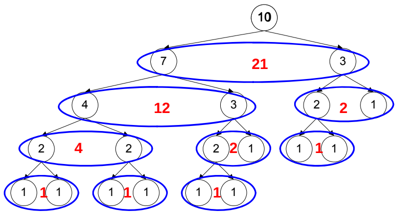
- Strong mathematical induction proves P(0), and assumes P(0), P(1), …, P(k) are all true, and uses that to show that P(k+1) is true.

$$[\mathrm{P}(0) \wedge \mathrm{P}(1) \wedge \mathrm{P}(2) \wedge \ldots \wedge \mathrm{P}(k)] \rightarrow \mathrm{P}(k+1)$$

Inducting stones

- Take a pile of n stones
  - Split the pile into two smaller piles of size r and s
  - Repeat until you have n piles of 1 stone each
- Take the product of **all** the splits
  - So all the r's and s's from **each** split
- Sum up each of these products
- Prove that this product, f(n), equals $\frac{n(n-1)}{2}$

Inducting stones



$$\frac{n(n-1)}{2}$$

$$f(n) = 21+12+2+4+2+1+1+1+1 = 45 = \frac{10*9}{2}$$

Inducting stones

- We will show by induction that P(*n*) is true for all *n* > 0, where P(*n*): f(*n*) = *n*(*n*+1)/2.

- Base case: *n* = 1
  - No splits necessary, so the sum of the products = 0
  - f(1) = 1*(1-1)/2 = 0
  - Base case proven

Inducting stones

- Inductive hypothesis: assume that P(1), P(2), …, P(*k*) are all true
  - This is strong induction!

- Inductive step: Show that P(*k*+1) is true
  - We assume that we split the *k*+1 pile into a pile of *i* stones and a pile of *k*+1-*i* stones
  - Thus, we want to show that (*i*)*(*k*+1-*i*) + f(*i*) + f(*k*+1-*i*) = (*k*+1)((k+1)-1)/2
  - Since 0 < *i* < *k+1*, both *i* and *k*+1-*i* are between 1 and *k*, inclusive

Inducting stones

Thus, we want to show that
(*i*)*(*k*+1-*i*) + f(*i*) + f(*k*+1-*i*) = (*k*+1)k/2

$$\mathrm{P}(i): f(i) = \frac{i^2 - i}{2}$$

$$\mathrm{P}(k+1-i): f(k+1-i) = \frac{(k+1-i)(k+1-i-1)}{2} = \frac{k^2+k-2ki-i+i^2}{2}$$

We prove $\mathrm{P}(k+1): f(k+1) = \frac{(k+1)(k+1-1)}{2} = \frac{k^2+k}{2}$

Because $f(k+1) = (i)*(k+1-i) + f(i) + f(k+1-i)$

$$ki + i - i^2 + \frac{i^2-i}{2} + \frac{k^2+k-2ki-i-i^2}{2} = \frac{k^2+k}{2}$$

$$2ki + 2i - 2i^2 + i^2 - i + k^2 + k - 2ki - i + i^2 = k^2 + k$$

$$k^2 + k = k^2 + k$$

Typical Induction Errors

- For all positive integers n, $3^n – 2$ is even.
- P(n): $3^n – 2$ is even.
- Induction hypothesis: Assume P(n) is true
- Inductive case: We want to show P(n+1) is true. Because $3^{n+1} – 2 = 3*3n – 2 = 2*3^n + (3^n – 2)$,
    - $2*3^n$ is even
    - $(3^n – 2)$ is even by Induction hypothesis
- So $3^{n+1} – 2$ is even.

Typical Induction Errors

- All positive integers are odd.
- P(n): n is odd
- Base case: P(1) is true
- Induction hypothesis: Assume P(1), P(2), …, P(n-2), P(n-1) are true
- Inductive case: P(n-2) is true means n-2 is odd. So is (n-2)+2 = n. Hence P(n) is true.

Typical Induction Errors

- All horses are the same color.
- P(n): n horses are the same color
- Base case: P(1) is true
- Induction hypothesis: Assume P(n-1) is true
- Inductive case: Suppose we have n horses, they are $a_1, a_2, a_3, \ldots, a_n$. Since both $\{a_1, a_2, a_3, \ldots, a_{n-1}\}$ and $\{a_2, a_3, \ldots, a_n\}$ have n-1 horses, they have the same color. Since $a_2, a_3, \ldots, a_{n-1}$ are common in both sets, so all have the same colors.

Typical Induction Errors

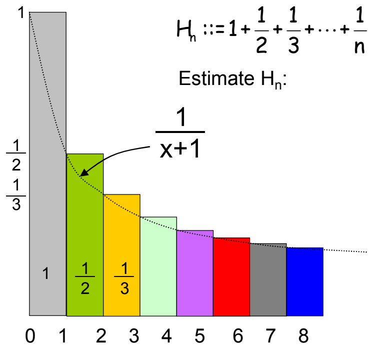
- A camel can always carry all the straw in a barn.
- P(n): camel can carry n straws
- Base case: P(1) is true
- Induction hypothesis: Assume P(n-1) is true
- Inductive case: Since the camel can carry n-1 straws, it has no problem to carry one more straw.

- “The last straw that broke the camel’s back”

Harmonic Number

$$H_n ::= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Estimate H_n:



Integral Method

$$\int_0^n \frac{1}{x+1}\,dx \le 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

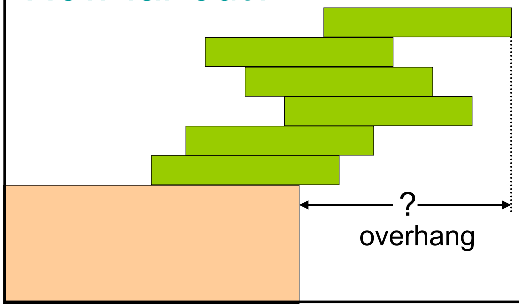
$$\int_1^{n+1} \frac{1}{x}\,dx \le H_n$$

$$\ln(n+1) \le H_n$$

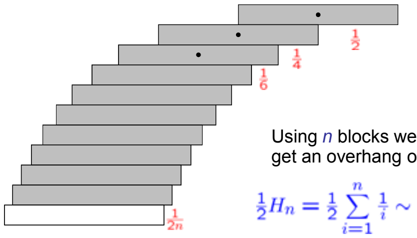
Now $H_n \to \infty$ as $n \to \infty$, so
Harmonic series can go to infinity!

Book Stacking

How far out?



The classical solution



Using *n* blocks we can get an overhang of

$$\tfrac{1}{2}H_n = \tfrac{1}{2}\sum_{i=1}^{n}\tfrac{1}{i} \sim \tfrac{1}{2}\ln n$$

Harmonic Stacks

Product

$$\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdots a_n$$

$$\prod_{k=1}^{5} k^2 \quad = (5!)^2$$

$$\prod_{k=1}^{n} \frac{k}{k+1} \quad = \frac{1}{n+1}$$

$$\prod_{k=1}^{n} 2^k \quad = 2^{n(n+1)/2}$$

Factorial

Factorial defines a product:

$$n! = 1 \cdot 2 \cdot 3 \cdot n = \prod_{i=1}^{n} i$$

How to estimate n!?

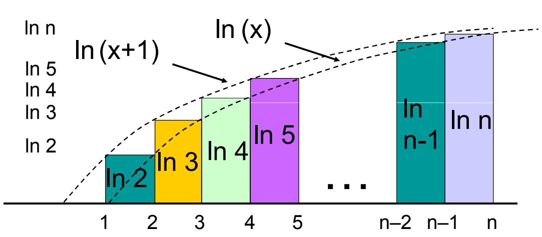
Turn product into a sum taking logs:

ln(n!) = ln(1·2·3 ··· (n – 1)·n)

= ln 1 + ln 2 + ··· + ln(n – 1) + ln(n)

$$= \sum_{i=1}^{n} \ln(i)$$

Integral Method



Analysis (OPTIONAL)

$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^n \ln(i) \le \int_0^n \ln(x+1)dx$$

Reminder: $\int \ln x\, dx = x \ln\left(\frac{x}{e}\right)$

$$n \ln(n/e) + 1 \le \sum \ln(i) \le (n+1) \ln((n+1)/e) + 1$$

so guess: $\sum_{i=1}^n \ln(i) \approx \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)$

Stirling’s Formula

$$\sum_{i=1}^n \ln(i) \approx \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)$$

exponentiating: $n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^n$

Stirling’s formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$