Sequences and Summations

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Chapter 4
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Definitions

- Sequence: an ordered list of elements
    - Elements can be duplicated
    - Elements are ordered

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Sequences

- A sequence is often given as
    - $a_1, a_2, \ldots, a_n, \ldots$
    - a_n is a term in the sequence.
- A sequence is actually a function *f* from a subset of **Z** to a set S
    - Usually from the positive or non-negative integers
    - a_n is the image of n: $f(n) = a_n$

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Sequence examples

- $a_n = 3n$
  - The terms in the sequence are a_1, a_2, a_3, …
  - The sequence $\{a_n\}$ is { 3, 6, 9, 12, … }
- $b_n = 2^n$
  - The terms in the sequence are b_1, b_2, b_3, …
  - The sequence $\{b_n\}$ is { 2, 4, 8, 16, 32, … }
- Note that sequences are indexed from 1
  - Not in all other textbooks, though!

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Geometric vs. arithmetic sequences

- The difference is in how they grow
- Arithmetic sequences increase by a constant *amount*
  - $a_n = 3n$
  - The sequence is { 3, 6, 9, 12, … }
  - Each number is 3 more than the last
  - Of the form: $f(x) = dx + a$
- Geometric sequences increase by a constant *factor*
  - $b_n = 2^n$
  - The sequence is { 2, 4, 8, 16, 32, … }
  - Each number is twice the previous
  - Of the form: $f(x) = ar^x$

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Fibonacci sequence

- Sequences can be neither geometric or arithmetic
  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence
  - Full formula: $F(n) = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5}\cdot 2^n}$

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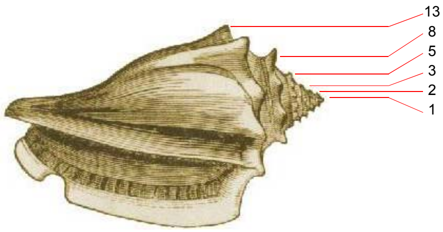
Fibonacci sequence in nature



Reproducing rabbits

- You have one pair of rabbits on an island
  - The rabbits repeat the following:
    - Get pregnant one month
    - Give birth to another pair the next month
    - Recover after giving birth for one month
  - This process repeats indefinitely (no deaths)
  - Rabbits get pregnant the month they are born

- How many rabbits are there after 10 months?

Reproducing rabbits

- First month: 1 pair
  - The original pair
- Second month: 1 pair
  - The original (and now pregnant) pair
- Third month: 2 pairs
  - The child pair (which is pregnant) and the parent pair (recovering)
- Fourth month: 3 pairs
  - "Grandchildren": Children from the baby pair (now pregnant)
  - Child pair (recovering)
  - Parent pair (pregnant)
- Fifth month: 5 pairs
  - Both the grandchildren and the parents reproduced
  - 3 pairs are pregnant (child and the two new born rabbits)

Reproducing rabbits

- Sixth month: 8 pairs
  - All 3 new rabbit pairs are pregnant, as well as those not pregnant in the last month (2)
- Seventh month: 13 pairs
  - All 5 new rabbit pairs are pregnant, as well as those not pregnant in the last month (3)
- Eighth month: 21 pairs
  - All 8 new rabbit pairs are pregnant, as well as those not pregnant in the last month (5)
- Ninth month: 34 pairs
  - All 13 new rabbit pairs are pregnant, as well as those not pregnant in the last month (8)
- Tenth month: 55 pairs
  - All 21 new rabbit pairs are pregnant, as well as those not pregnant in the last month (13)

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Reproducing rabbits

- Note the sequence:

  { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }

- The Fibonacci sequence again

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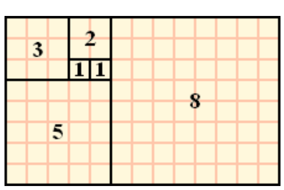
Fibonacci sequence

- Another application:



- Fibonacci references from http://en.wikipedia.org/wiki/Fibonacci_sequence

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Fibonacci sequence

- As the terms increase, the ratio between successive terms approaches 1.618

$$\lim_{n\to\infty}\frac{F(n+1)}{F(n)}=\phi=\frac{\sqrt{5}+1}{2}=1.618933989$$

- This is called the “golden ratio”
  - Ratio of human leg length to arm length
  - Ratio of successive layers in a conch shell

- Reference: http://en.wikipedia.org/wiki/Golden_ratio

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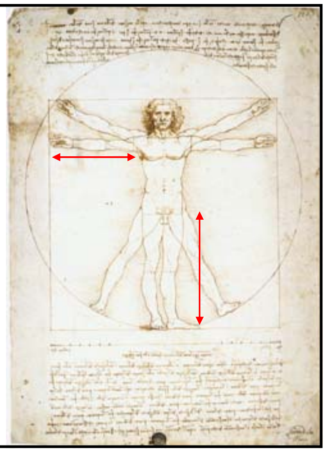
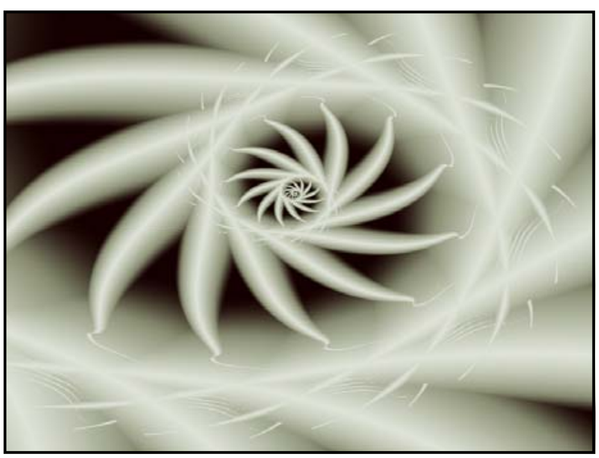
The Golden Ratio





Determining the sequence formula

- Given values in a sequence, how do you determine the formula?
- Steps to consider:
  - Is it an arithmetic progression (each term a constant amount from the last)?
  - Is it a geometric progression (each term a factor of the previous term)?
  - Does the sequence repeat (or cycle)?
  - Does the sequence combine previous terms?
  - Are there runs of the same value?

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Determining the sequence formula

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …
  - The sequence alternates 1's and 0's, increasing the number of 1's and 0's each time

b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …
  - This sequence increases by one, but repeats all even numbers once

c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …
  - The non-0 numbers are a geometric sequence (2^n) interspersed with zeros

d) 3, 6, 12, 24, 48, 96, 192, …
  - Each term is twice the previous: geometric progression
  - $a_n = 3*2^{n-1}$

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Determining the sequence formula

e) 15, 8, 1, -6, -13, -20, -27, …
  - Each term is 7 less than the previous term
  - $a_n = 22 - 7n$

f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …
  - The difference between successive terms increases by one each time
  - $a_1 = 3$, $a_n = a_{n-1} + n$
  - $a_n = n(n+1)/2 + 2$

g) 2, 16, 54, 128, 250, 432, 686, …
  - Each term is twice the cube of *n*
  - $a_n = 2*n^3$

h) 2, 3, 7, 25, 121, 721, 5041, 40321
  - Each successive term is about *n* times the previous
  - $a_n = n! + 1$
  - My solution: $a_n = a_{n-1} * n - n + 1$

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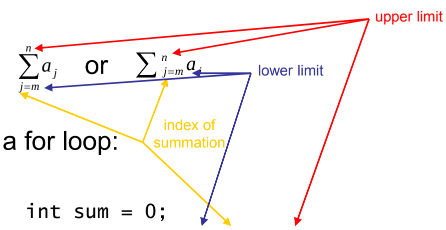
Useful sequences

- n^2 = 1, 4, 9, 16, 25, 36, …
- n^3 = 1, 8, 27, 64, 125, 216, …
- n^4 = 1, 16, 81, 256, 625, 1296, …
- 2^n = 2, 4, 8, 16, 32, 64, …
- 3^n = 3, 9, 27, 81, 243, 729, …
- $n!$ = 1, 2, 6, 24, 120, 720, …

Summations

- A summation:

$\sum_{j=m}^{n} a_j$ or $\sum_{j=m}^{n} a_j$



- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
     sum += a(j);
```


Evaluating sequences

$\sum_{k=1}^{5}(k+1)$

- 2 + 3 + 4 + 5 + 6 = 20

$\sum_{j=0}^{4}(-2)^j$

- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$\sum_{i=1}^{10} 3$

- 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30

$\sum_{j=0}^{8}\left(2^{j+1} - 2^j\right)$

- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^{10}-2^9) = 511$
  - Note that each term (except the first and last) is cancelled by another term

Evaluating sequences

- Let S = { 1, 3, 5, 7 }
- What is $\Sigma_{j \in S}\, j$
  - 1 + 3 + 5 + 7 = 16
- What is $\Sigma_{j \in S}\, j^2$
  - $1^2 + 3^2 + 5^2 + 7^2 = 84$
- What is $\Sigma_{j \in S}\, (1/j)$
  - 1/1 + 1/3 + 1/5 + 1/7 = 176/105
- What is $\Sigma_{j \in S}\, 1$
  - 1 + 1 + 1 + 1 = 4

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Summation

$$\sum_{i=1}^{n} a_i := a_1 + a_2 + \cdots + a_n$$

$$\sum_{k=1}^{5} k^2$$

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$$

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

$$\sum_{i=1}^{n} \frac{1}{i^2} = \sum_{i=1}^{n-1} \frac{1}{i^2} + \frac{1}{n^2}?$$

$$\sum_{k=0}^{n-1} 2^k + 2^n = \sum_{k=0}^{n} 2^k?$$

A Telescoping Sum

$$\sum_{i=1}^{n} \frac{1}{k(k+1)}$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

When do we have closed form formulas?

Sum for Children

89 + 102 + 115 + 128 + 141 +
154 + ··· +
193 + ··· +
232 + ··· +
323 + ··· +
414 + ··· + 453 + 466

Nine-year old Gauss saw
30 numbers, each 13 greater than the previous one.

$1^{st} + 30^{th} = 89 + 466 = 555$

$2^{nd} + 29^{th} = (1^{st}+13) + (30^{th}-13) = 555$

$3^{rd} + 28^{th} = (2^{nd}+13) + (29^{th}-13) = 555$

So the sum is equal to 15x555 = 8325.

Arithmetic Series

Given n numbers, $a_1, a_2, \ldots, a_n$ with common difference d, i.e. $a_{i+1} - a_i = d$.

What is a simple **closed form** expression of the sum? $S_n = \sum_{i=1}^{n} a_i$

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots\ldots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$$

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-2)d) + (a_n - (n-1)d)$$

Adding the equations together gives:

$$2S_n = n(a_1 + a_n)$$

Rearranging and remembering that $a_n = a_1 + (n-1)d$, we get:

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n[2a_1 + (n-1)d]}{2}$$

Geometric Series

$$G_n ::= 1 + x + x^2 + \cdots + x^{n-1} + x^n$$

What is the closed form expression of G_n? $G_n = \sum_{i=0}^{n} x^i$

$$G_n ::= 1 + x + x^2 + \cdots + x^{n-1} + x^n$$

$$xG_n = x + x^2 + x^3 + \cdots + x^n + x^{n+1}$$

$$G_n - xG_n = 1 - x^{n+1}$$

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

Infinite Geometric Series

$$G_n = \frac{1-x^{n+1}}{1-x}$$

Consider *infinite* sum (series)

$$1+x+x^2+\cdots+x^{n-1}+x^n+\cdots = \sum_{i=0}^{\infty} x^i$$

$$\lim_{n\to\infty} G_n = \frac{1-\lim_{n\to\infty} x^{n+1}}{1-x} = \frac{1}{1-x}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$ for $|x| < 1$

Some Examples

$$1+1/2+1/4+1/8+\cdots = \sum_{i=0}^{\infty}(1/2)^i$$

$$0.999999999\ldots = 0.9\sum_{i=0}^{\infty}(1/10)^i$$

$$1-1/2+1/4-1/8+\cdots = \sum_{i=0}^{\infty}(-1/2)^i$$

$$1+2+4+8+\cdots+2^{n-1} = \sum_{i=0}^{n-1} 2^i$$

$$1+3+9+27+\cdots+3^{n-1} = \sum_{i=0}^{n-1} 3^i$$

Double summations

- Like a nested for loop

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
     for ( int j = 1; j <= 3; j++ )
          sum += i*j;
```

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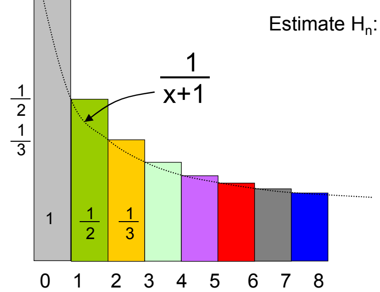
Harmonic Number

$$H_n ::= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Estimate H_n:



Integral Method

$$\int_0^n \frac{1}{x+1}\,dx \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

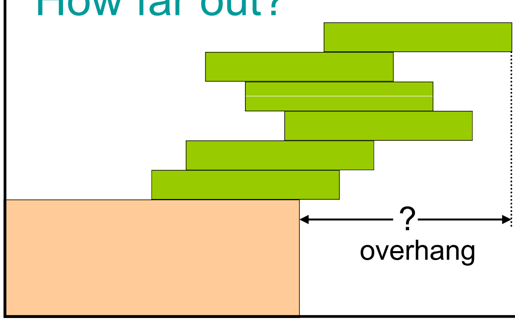
$$\int_1^{n+1} \frac{1}{x}\,dx \le H_n$$

$$\ln(n+1) \le H_n$$

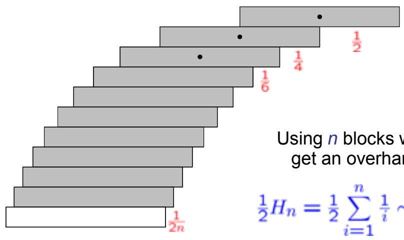
Now $H_n \to \infty$ as $n \to \infty$, so
Harmonic series can go to infinity!

Book Stacking

How far out?



The classical solution



Using n blocks we can get an overhang of

$$\frac{1}{2}H_n = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{i} \sim \frac{1}{2}\ln n$$

Harmonic Stacks

Product

$$\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdots a_n$$

$$\prod_{k=1}^{5} k^2$$

$$\prod_{k=1}^{n} \frac{k}{k+1}$$

$$\prod_{k=1}^{n} 2^k$$

Factorial

Factorial defines a product:

$$n! = 1 \cdot 2 \cdot 3 \cdot n = \prod_{i=1}^{n} i$$

How to estimate n!?

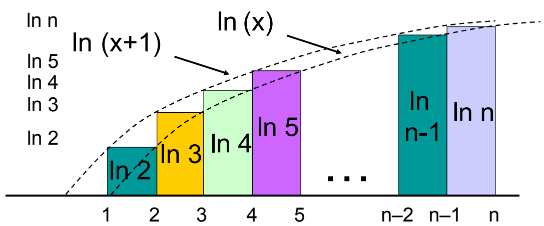
Turn product into a sum taking logs:

ln(n!) = ln(1·2·3 ··· (n – 1)·n)

= ln 1 + ln 2 + ··· + ln(n – 1) + ln(n)

$$= \sum_{i=1}^{n} \ln(i)$$

Integral Method



Analysis (OPTIONAL)

$$\int_1^n \ln(x)\,dx \leq \sum_{i=1}^n \ln(i) \leq \int_0^n \ln(x+1)dx$$

Reminder: $\int \ln x \, dx = x \ln\left(\frac{x}{e}\right)$

$$n \ln(n/e) + 1 \leq \sum \ln(i) \leq (n+1)\ln((n+1)/e) + 1$$

so guess: $\sum_{i=1}^{n} \ln(i) \approx \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)$

Stirling's Formula

$$\sum_{i=1}^{n} \ln(i) \approx \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)$$

exponentiating: $n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^n$

Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$