

Predicates and Quantifiers

22c:19, Chapter 2
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Terminology review

- Proposition: a statement that is either true or false
 - Must always be one or the other!
 - Example: “The sky is red”
 - Not a proposition: $x + 3 > 4$
- Boolean variable: A variable (usually p , q , r , etc.) that represents a proposition

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Propositional functions

- Consider $P(x) = x < 5$
 - $P(x)$ has no truth values (x is not given a value)
 - $P(1)$ is true
 - The proposition $1 < 5$ is true
 - $P(10)$ is false
 - The proposition $10 < 5$ is false
- Thus, $P(x)$ will create a proposition when given a value

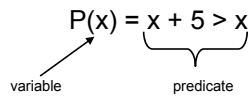
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Propositional functions

- Let $P(x) = \text{"}x \text{ is a multiple of } 5\text{"}$
 - For what values of x is $P(x)$ true?
- Let $P(x) = x + 1 > x$
 - For what values of x is $P(x)$ true?
- Let $P(x) = x + 3$
 - For what values of x is $P(x)$ true?

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Anatomy of a propositional function



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Propositional functions

- Functions with multiple variables:
 - $P(x,y) = x + y == 0$
 - $P(1,2)$ is false, $P(1,-1)$ is true
 - $P(x,y,z) = x + y == z$
 - $P(3,4,5)$ is false, $P(1,2,3)$ is true
 - $P(x_1, x_2, x_3 \dots x_n) = \dots$

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So, why do we care about quantifiers?

- Many things (in this course and beyond) are specified using quantifiers
 - In some cases, it's a more accurate way to describe things than Boolean propositions

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Quantifiers

- A quantifier is “an operator that limits the variables of a proposition”
- Two types:
 - Universal
 - Existential

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Universal quantifiers

- Represented by an upside-down A: \forall
 - It means “for all”
 - Let $P(x) = x+1 > x$
- We can state the following:
 - $\forall x P(x)$
 - English translation: “for all values of x , $P(x)$ is true”
 - English translation: “for all values of x , $x+1 > x$ is true”

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Universal quantifiers

- But is that always true?
 - $\forall x P(x)$
- Let $x =$ the character 'a'
 - Is 'a'+1 > 'a'?
- Let $x =$ the state of Virginia
 - Is Virginia+1 > Virginia?
- You need to specify your universe!
 - What values x can represent
 - Called the "domain" or "universe of discourse" by the textbook

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Universal quantifiers

- Let the universe be the real numbers.
 - Then, $\forall x P(x)$ is true
- Let $P(x) = x/2 < x$
 - Not true for the negative numbers!
 - Thus, $\forall x P(x)$ is false
 - When the domain is all the real numbers
- In order to prove that a universal quantification is true, it must be shown for ALL cases
- In order to prove that a universal quantification is false, it must be shown to be false for only ONE case

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Universal quantification

- Given some propositional function $P(x)$
- And values in the universe $x_1 \dots x_n$
- The universal quantification $\forall x P(x)$ implies:

$$P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$$

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Universal quantification

- Think of \forall as a for loop:
- $\forall x P(x)$, where $1 \leq x \leq 10$
- ... can be translated as ...


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for ( x = 1; x <= 10; x++ )
  is P(x) true?
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- If $P(x)$ is true for all parts of the for loop, then $\forall x P(x)$
 - Consequently, if $P(x)$ is false for any one value of the for loop, then $\forall x P(x)$ is false

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Existential quantification

- Represented by an backwards E: \exists
 - It means “there exists”
 - Let $P(x) = x+1 > x$
- We can state the following:
 - $\exists x P(x)$
 - English translation: “there exists (a value of) x such that $P(x)$ is true”
 - English translation: “for at least one value of x , $x+1 > x$ is true”

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Existential quantification

- Note that you still have to specify your universe
 - If the universe we are talking about is all the states in the US, then $\exists x P(x)$ is not true
- Let $P(x) = x+1 < x$
 - There is no numerical value x for which $x+1 < x$
 - Thus, $\exists x P(x)$ is false

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Existential quantification

- Let $P(x) = x+1 > x$
 - There is a numerical value for which $x+1 > x$
 - In fact, it's true for all of the values of x !
 - Thus, $\exists x P(x)$ is true
- In order to show an existential quantification is true, you only have to find ONE value
- In order to show an existential quantification is false, you have to show it's false for ALL values

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Existential quantification

- Given some propositional function $P(x)$
- And values in the universe $x_1 \dots x_n$
- The existential quantification $\exists x P(x)$ implies:

$$P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$$

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A note on quantifiers

- Recall that $P(x)$ is a propositional function
 - Let $P(x)$ be " $x == 0$ "
- Recall that a proposition is a statement that is either true or false
 - $P(x)$ is not a proposition
- There are two ways to make a propositional function into a proposition:
 - Supply it with a value
 - For example, $P(5)$ is false, $P(0)$ is true
 - Provide a quantification
 - For example, $\forall x P(x)$ is false and $\exists x P(x)$ is true
 - Let the universe of discourse be the real numbers

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Binding variables

- Let $P(x,y)$ be $x > y$
- Consider: $\forall x P(x,y)$
 - This is not a proposition!
 - What is y ?
 - If it's 5, then $\forall x P(x,y)$ is false
 - If it's $x-1$, then $\forall x P(x,y)$ is true
- Note that y is not “bound” by a quantifier

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Binding variables

- $(\exists x P(x)) \vee Q(x)$
 - The x in $Q(x)$ is not bound; thus not a proposition
- $(\exists x P(x)) \vee (\forall x Q(x))$
 - Both x values are bound; thus it is a proposition
- $(\exists x P(x) \wedge Q(x)) \vee (\forall y R(y))$
 - All variables are bound; thus it is a proposition
- $(\exists x P(x) \wedge Q(y)) \vee (\forall y R(y))$
 - The y in $Q(y)$ is not bound; this not a proposition

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Negating quantifications

- Consider the statement:
 - All students in this class have red hair
- What is required to show the statement is false?
 - There exists a student in this class that does NOT have red hair
- To negate a universal quantification:
 - You negate the propositional function
 - AND you change to an existential quantification
 - $\neg \forall x P(x) = \exists x \neg P(x)$

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Negating quantifications

- Consider the statement:
 - There is a student in this class with red hair
- What is required to show the statement is false?
 - All students in this class do not have red hair
- Thus, to negate an existential quantification:
 - You negate the propositional function
 - AND you change to a universal quantification
 - $\neg \exists x P(x) = \forall x \neg P(x)$

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Translating from English

- Consider “For every student in this class, that student has studied calculus”
- Rephrased: “For every student x in this class, x has studied calculus”
 - Let $C(x)$ be “ x has studied calculus”
 - Let $S(x)$ be “ x is a student”
- $\forall x C(x)$
 - True if the universe of discourse is all students in this class

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Translating from English

- What about if the universe of discourse is all students (or all people?)
 - $\forall x (S(x) \wedge C(x))$
 - This is wrong! Why?
 - $\forall x (S(x) \rightarrow C(x))$
- Another option:
 - Let $Q(x,y)$ be “ x has studied y ”
 - $\forall x (S(x) \rightarrow Q(x, \text{calculus}))$

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Translating from English

- Consider:
 - “Some students have visited Mexico”
 - “Every student in this class has visited Canada or Mexico”
- Let:
 - $S(x)$ be “ x is a student in this class”
 - $M(x)$ be “ x has visited Mexico”
 - $C(x)$ be “ x has visited Canada”

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Translating from English

- Consider: “Some students have visited Mexico”
 - Rephrasing: “There exists a student who has visited Mexico”
- $\exists x M(x)$
 - True if the universe of discourse is all students
- What about if the universe of discourse is all people?
 - $\exists x (S(x) \rightarrow M(x))$
 - This is wrong! Why?
 - $\exists x (S(x) \wedge M(x))$

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Translating from English

- Consider: “Every student in this class has visited Canada or Mexico”
- $\forall x (M(x) \vee C(x))$
 - When the universe of discourse is all students
- $\forall x (S(x) \rightarrow (M(x) \vee C(x)))$
 - When the universe of discourse is all people
- Why isn't $\forall x (S(x) \wedge (M(x) \vee C(x)))$ correct?

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Translating from English

- Note that it would be easier to define $V(x, y)$ as “x has visited y”
 - $\forall x (S(x) \wedge V(x, \text{Mexico}))$
 - $\forall x (S(x) \rightarrow (V(x, \text{Mexico}) \vee V(x, \text{Canada})))$

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Translating from English

- Translate the statements:
 - “All hummingbirds are richly colored”
 - “No large birds live on honey”
 - “Birds that do not live on honey are dull in color”
 - “Hummingbirds are small”
- Assign our propositional functions
 - Let $P(x)$ be “x is a hummingbird”
 - Let $Q(x)$ be “x is large”
 - Let $R(x)$ be “x lives on honey”
 - Let $S(x)$ be “x is richly colored”
- Let our universe of discourse be all birds

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Translating from English

- Our propositional functions
 - Let $P(x)$ be “x is a hummingbird”
 - Let $Q(x)$ be “x is large”
 - Let $R(x)$ be “x lives on honey”
 - Let $S(x)$ be “x is richly colored”
- Translate the statements:
 - “All hummingbirds are richly colored”
 - $\forall x (P(x) \rightarrow S(x))$
 - “No large birds live on honey”
 - $\neg \exists x (Q(x) \wedge R(x))$
 - Alternatively: $\forall x (\neg Q(x) \vee \neg R(x))$
 - “Birds that do not live on honey are dull in color”
 - $\forall x (\neg R(x) \rightarrow \neg S(x))$
 - “Hummingbirds are small”
 - $\forall x (P(x) \rightarrow \neg Q(x))$

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Prolog

- A programming language using logic!
- Entering facts:
instructor (bloomfield, cs202)
enrolled (alice, cs202)
enrolled (bob, cs202)
enrolled (claire, cs202)
- Entering predicates:
teaches (P,S) :- instructor (P,C), enrolled (S,C)
- Extracting data
?enrolled (alice, cs202)
 - Result:
yes

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Prolog

- Extracting data
?enrolled (X, cs202)
 - Result:
alice
bob
claire
- Extracting data
?teaches (X, alice)
 - Result:
bloomfield

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Multiple quantifiers

- You can have multiple quantifiers on a statement
- $\forall x \exists y P(x, y)$
 - "For all x, there exists a y such that P(x,y)"
 - Example: $\forall x \exists y (x+y == 0)$
- $\exists x \forall y P(x,y)$
 - There exists an x such that for all y P(x,y) is true"
 - Example: $\exists x \forall y (x*y == 0)$

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Order of quantifiers

- $\exists x \forall y$ and $\forall x \exists y$ are not equivalent!
- $\exists x \forall y P(x,y)$
 - $P(x,y) = (x+y == 0)$ is false
- $\forall x \exists y P(x,y)$
 - $P(x,y) = (x+y == 0)$ is true

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Negating multiple quantifiers

- Recall negation rules for single quantifiers:
 - $\neg \forall x P(x) = \exists x \neg P(x)$
 - $\neg \exists x P(x) = \forall x \neg P(x)$
 - Essentially, you change the quantifier(s), and negate what it's quantifying
- Examples:
 - $\neg(\forall x \exists y P(x,y))$
 - = $\exists x \neg \exists y P(x,y)$
 - = $\exists x \forall y \neg P(x,y)$
 - $\neg(\forall x \exists y \forall z P(x,y,z))$
 - = $\exists x \neg \exists y \forall z P(x,y,z)$
 - = $\exists x \forall y \neg \forall z P(x,y,z)$
 - = $\exists x \forall y \exists z \neg P(x,y,z)$

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Negating multiple quantifiers

- Consider $\neg(\forall x \exists y P(x,y)) = \exists x \forall y \neg P(x,y)$
 - The left side is saying "for all x, there exists a y such that P is true"
 - To disprove it (negate it), you need to show that "there exists an x such that for all y, P is false"
- Consider $\neg(\exists x \forall y P(x,y)) = \forall x \exists y \neg P(x,y)$
 - The left side is saying "there exists an x such that for all y, P is true"
 - To disprove it (negate it), you need to show that "for all x, there exists a y such that P is false"

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Translating between English and quantifiers

- The product of two negative integers is positive
 - $\forall x \forall y ((x < 0) \wedge (y < 0) \rightarrow (xy > 0))$
 - Why conditional instead of and?
- The average of two positive integers is positive
 - $\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow ((x+y)/2 > 0))$
- The difference of two negative integers is not necessarily negative
 - $\exists x \exists y ((x < 0) \wedge (y < 0) \wedge (x-y \geq 0))$
 - Why and instead of conditional?
- The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers
 - $\forall x \forall y (|x+y| \leq |x| + |y|)$

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Translating between English and quantifiers

- $\exists x \forall y (x+y = y)$
 - There exists an additive identity for all real numbers
- $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (x-y > 0))$
 - A non-negative number minus a negative number is greater than zero
- $\exists x \exists y (((x \leq 0) \wedge (y \leq 0)) \wedge (x-y > 0))$
 - The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)
- $\forall x \forall y (((x \neq 0) \wedge (y \neq 0)) \leftrightarrow (xy \neq 0))$
 - The product of two non-zero numbers is non-zero if and only if both factors are non-zero

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Negation examples

- Rewrite these statements so that the negations only appear within the predicates
- $\neg \exists y \exists x P(x,y)$
 - $\forall y \neg \exists x P(x,y)$
 - $\forall y \forall x \neg P(x,y)$
 - $\neg \forall x \exists y P(x,y)$
 - $\exists x \neg \exists y P(x,y)$
 - $\exists x \forall y \neg P(x,y)$
 - $\neg \exists y (Q(y) \wedge \forall x \neg R(x,y))$
 - $\forall y \neg (Q(y) \wedge \forall x \neg R(x,y))$
 - $\forall y (\neg Q(y) \vee \neg (\forall x \neg R(x,y)))$
 - $\forall y (\neg Q(y) \vee \exists x R(x,y))$

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Negation examples

- Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z T(x,y,z)$

- $\neg(\forall x \exists y \forall z T(x,y,z))$
- $\neg \forall x \exists y \forall z T(x,y,z)$
- $\exists x \neg \exists y \forall z T(x,y,z)$
- $\exists x \forall y \neg \forall z T(x,y,z)$
- $\exists x \forall y \exists z \neg T(x,y,z)$

b) $\forall x \exists y P(x,y) \vee \forall x \exists y Q(x,y)$

- $\neg(\forall x \exists y P(x,y) \vee \forall x \exists y Q(x,y))$
- $\neg \forall x \exists y P(x,y) \wedge \neg \forall x \exists y Q(x,y)$
- $\exists x \neg \exists y P(x,y) \wedge \exists x \neg \exists y Q(x,y)$
- $\exists x \forall y \neg P(x,y) \wedge \exists x \forall y \neg Q(x,y)$

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Rules of inference for the universal quantifier

- Assume that we know that $\forall x P(x)$ is true
 - Then we can conclude that $P(c)$ is true
 - Here c stands for some specific constant
 - This is called “universal instantiation”
- Assume that we know that $P(c)$ is true for any value of c
 - Then we can conclude that $\forall x P(x)$ is true
 - This is called “universal generalization”

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Rules of inference for the existential quantifier

- Assume that we know that $\exists x P(x)$ is true
 - Then we can conclude that $P(c)$ is true for some value of c
 - This is called “existential instantiation”
- Assume that we know that $P(c)$ is true for some value of c
 - Then we can conclude that $\exists x P(x)$ is true
 - This is called “existential generalization”

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Example of proof

- Given the hypotheses:
 - “Linda, a student in this class, owns a red convertible.” $C(Linda)$
 $R(Linda)$
 - “Everybody who owns a red convertible has gotten at least one speeding ticket” $\forall x (R(x) \rightarrow T(x))$
- Can you conclude: “Somebody in this class has gotten a speeding ticket”? $\exists x (C(x) \wedge T(x))$

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Example of proof

1. $\forall x (R(x) \rightarrow T(x))$ 3rd hypothesis
2. $R(Linda) \rightarrow T(Linda)$ Universal instantiation using step 1
3. $R(Linda)$ 2nd hypothesis
4. $T(Linda)$ Modes ponens using steps 2 & 3
5. $C(Linda)$ 1st hypothesis
6. $C(Linda) \wedge T(Linda)$ Conjunction using steps 4 & 5
7. $\exists x (C(x) \wedge T(x))$ Existential generalization using step 6

Thus, we have shown that “Somebody in this class has gotten a speeding ticket”

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Example of proof

- Given the hypotheses:
 - “There is someone in this class who has been to France” $\exists x (C(x) \wedge F(x))$
 - “Everyone who goes to France visits the Louvre” $\forall x (F(x) \rightarrow L(x))$
- Can you conclude: “Someone in this class has visited the Louvre”? $\exists x (C(x) \wedge L(x))$

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Example of proof

1. $\exists x (C(x) \wedge F(x))$ 1st hypothesis
2. $C(y) \wedge F(y)$ Existential instantiation using step 1
3. $F(y)$ Simplification using step 2
4. $C(y)$ Simplification using step 2
5. $\forall x (F(x) \rightarrow L(x))$ 2nd hypothesis
6. $F(y) \rightarrow L(y)$ Universal instantiation using step 5
7. $L(y)$ Modus ponens using steps 3 & 6
8. $C(y) \wedge L(y)$ Conjunction using steps 4 & 7
9. $\exists x (C(x) \wedge L(x))$ Existential generalization using step 8

Thus, we have shown that "Someone
in this class has visited the Louvre"

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