Valid and Invalid Arguments

22c:19
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What is a Fallacy?

- A fallacy is an error in an argument.
- An argument consists of one or more premises and one conclusion.
- Both the premises (p_1, p_2, …, p_n) and the conclusion (q) are statements (i.e., true or false).
- If (p_1, p_2, …, $p_n \rightarrow q$) is not a tautology, then a fallacy occurs.
- If p_i is false, then a fallacy occurs.

Testing an Argument

- Identify the premises and conclusion.
- Make sure every premise is valid.
- Construct a true table for p_1, p_2, …, p_n, q and (p_1, p_2, …, $p_n \rightarrow q$).
- If there is one line such that (p_1, p_2, …, $p_n \rightarrow q$) is false, then the argument is invalid, or a fallacy.

Modus Ponens example

- Assume you are given the following two statements:
  - "you are in this class"
  - "if you are in this class, you will get a grade"

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

- Let p =: "you are in this class"
- Let q =: "you will get a grade"
- By Modus Ponens, you can conclude that you will get a grade.

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Modus Ponens

- Consider (p ∧ (p→q)) → q

p	q	p→q	p∧(p→q))	(p∧(p→q)) → q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

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Modus Tollens

- Assume that we know: ¬q and p → q
  - Recall that p → q ≡ ¬q → ¬p
- Thus, we know ¬q and ¬q → ¬p
- We can conclude ¬p

$$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$$

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Modus Tollens example

- Assume you are given the following two statements:
  - "you will not get a grade"
  - "if you are in this class, you will get a grade"
- Let p =: "you are in this class"
- Let q =: "you will get a grade"

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

- By Modus Tollens, you can conclude that you are not in this class

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Generalization & Specialization

- Generalization: If you know that p is true, then p ∨ q will ALWAYS be true

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

- Specialization: If p ∧ q is true, then p will ALWAYS be true

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

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Example of proof

- We have the hypotheses:
  - p q "It is not sunny this afternoon and it is colder than yesterday"
  - r "We will go swimming only if it is sunny"
  - s "If we do not go swimming, then we will take a canoe trip"
  - t "If we take a canoe trip, then we will be home by sunset"
- Does this imply that "we will be home by sunset"?

$$\begin{array}{l} \neg p \wedge q \\ r \rightarrow p \\ \neg r \rightarrow s \\ s \rightarrow t \\ \hline t \end{array}$$

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Example of proof

1. ¬p ∧ q — 1st hypothesis
2. ¬p — Simplification using step 1
3. r → p — 2nd hypothesis
4. ¬r — Modus tollens using steps 2 & 3
5. ¬r → s — 3rd hypothesis
6. s — Modus ponens using steps 4 & 5
7. s → t — 4th hypothesis
8. t — Modus ponens using steps 6 & 7

$$\frac{p \wedge q}{\therefore p} \qquad \frac{\begin{array}{c}p \\ p \rightarrow q\end{array}}{\therefore q} \qquad \frac{\begin{array}{c}\neg q \\ p \rightarrow q\end{array}}{\therefore \neg p}$$

So what did we show?

- We showed that:
  - ((¬p ∧ q) ∧ (r → p) ∧ (¬r → s) ∧ (s → t))→ t
  - That when the 4 hypotheses are true, then the implication is true
  - In other words, we showed the above is a tautology!

More rules of inference

- Conjunction: if p and q are true separately, then p∧q is true

$$\frac{\begin{array}{c}p \\ q\end{array}}{\therefore p \wedge q}$$

- Elimination: If p∨q is true, and p is false, then q must be true

$$\frac{\begin{array}{c}p \vee q \\ \neg p\end{array}}{\therefore q}$$

- Transitivity: If p→q is true, and q→r is true, then p→r must be true

$$\frac{\begin{array}{c}p \rightarrow q \\ q \rightarrow r\end{array}}{\therefore p \rightarrow r}$$

Even more rules of inference

- Proof by division into cases: if at least one of p or q is true, then r must be true

$$\begin{array}{l} p \vee q \\ p \to r \\ \underline{q \to r} \\ \therefore r \end{array}$$

- Contradiction rule: If ¬p→c is true, we can conclude p (via the contra-positive)

$$\begin{array}{l} \underline{\neg p \to c} \\ \therefore p \end{array}$$

- Resolution: If p∨q is true, and ¬p∨r is true, then q∨r must be true
  - Not in the textbook

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore q \vee r \end{array}$$

Example of proof

- Given the hypotheses:
  - "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on"
  - "If the sailing race is held, then the trophy will be awarded"
  - "The trophy was not awarded"
- Can you conclude: "It rained"?

(¬r ∨ ¬f) → (s ∧ l)

s → t

¬t

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r

Example of proof

1.	¬t	3rd hypothesis
2.	s → t	2nd hypothesis
3.	¬s	Modus tollens using steps 2 & 3
4.	(¬r∨¬f)→(s∧l)	1st hypothesis
5.	¬(s∧l)→¬(¬r∨¬f)	Contrapositive of step 4
6.	(¬s∨¬l)→(r∧f)	DeMorgan's law and double negation law
7.	¬s∨¬l	Addition from step 3
8.	r∧f	Modus ponens using steps 6 & 7
9.	r	Simplification using step 8

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore q \end{array} \qquad \begin{array}{l} \underline{p} \\ \therefore p \vee q \end{array} \qquad \begin{array}{l} \underline{p \wedge q} \\ \therefore p \end{array} \qquad \begin{array}{l} \neg q \\ \underline{p \to q} \\ \therefore \neg p \end{array}$$

Fallacy: Converse Error

- Consider the following:
- Is this true?

$$\begin{array}{l} q \\ \underline{p \to q} \\ \therefore p \end{array} \qquad \begin{array}{l} q \\ \underline{q \to p} \\ \therefore p \end{array}$$

p	q	p→q	q∧(p→q)	(q∧(p→q)) → p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Not a valid rule!

Fallacy: Inverse Error

- Consider the following:
- Is this true?

$$\begin{array}{l} \neg p \\ \underline{p \to q} \\ \therefore \neg q \end{array} \qquad \begin{array}{l} \neg p \\ \underline{\neg p \to \neg q} \\ \therefore \neg q \end{array}$$

p	q	p→q	¬p∧(p→q)	(¬p∧(p→q)) → ¬q
T	T	T	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

Not a valid rule!

Contradiction Rule

- If the supposition that statement p is false leads logically to a contradiction, then p is true.

$$\begin{array}{l} \underline{\neg p \to F} \\ \therefore p \end{array}$$

- Example: $\sqrt{2}$ is not rational.