Midterm I. (22C:19, Fall 2009)

Open books and notes; Total points = 100

1. Propositional Logic (20) Using the truth table to decide if the following statements are tautology, contradiction, or neither.

   (a) (10) $(p \wedge q) \rightarrow (p \rightarrow q)$

   (b) (10) $(\sim q \rightarrow \sim p) \rightarrow \sim (p \rightarrow q)$

   Answer:

   The truth value table is omitted. The first one is a tautology and the second one is neither tautology nor contradiction (it's satisfiable).

2. First-Order Logic (20) Let $P(x, y)$ denote the sentence: $x - y = 4$. Please state each of the following formulas in plain English and decide their truth values over the domain $\mathcal{Z}$:

   (a) (10) $\exists x \forall y P(x, y)$

   (b) (10) $\forall y \exists x \sim P(x, y)$

   Answer:

   $\exists x \forall y P(x, y)$ means "There exists an integer x such that for all integer y, $x - y = 4$". This is false because once x is fixed, there is only value, i.e., $y = x - 4$, to make $P(x, y)$ to be true.

   $\forall y \exists x \sim P(x, y)$ means "For any integer y, there exists an integer x such that $x - y \neq 4$". This is true because for any y, we may choose $x = y$ such that $x - y = 0 \neq 4$.

3. Integers and Computer Representation (20) Suppose an integer is represented by a 6-bit (including the sign bit) number in computer. Please compute the Two's complement of 10 and use it to illustrate how $9 + (-4)$ is computed in computer for 5-bit numbers.

   Answer:

   The 2's complement of 10 is $2^6 - 10 = 54 = (110110)_2$.

   The addition of 9 and -4 is implemented as the addition of 9 and 2's complement of 4. For 5-bit numbers, 9 is $(01001)_2$ and 2's complement of 4 is $(11110)_2$.

4. Logic Circuit (20)
   Please use a minimal number of AND, OR and NOT gates to construct a circuit for the following Boolean expression:

   $$(p \vee q) \rightarrow \ (\sim r \wedge \sim q \vee \sim r \wedge p)$$

   Answer:

	$(p \vee q) \rightarrow \ (\sim r \wedge \sim q \vee \sim r \wedge p)$	
$\equiv$	$\sim (p \vee q) \vee \ (\sim r \wedge \sim q \vee \sim r \wedge p)$	definition of $\rightarrow$
$\equiv$	$\sim (p \vee q) \vee \ (\sim r \ \wedge (\sim q \ \vee p))$	distribution of $\wedge$ over $\vee$
$\equiv$	$\sim (p \vee q) \vee \ (\sim r \wedge \sim (q \wedge \sim p))$	DeMogen law and double negation
$\equiv$	$\sim (p \vee q) \vee \ \ \sim (r \ \vee q \wedge \sim p)$	DeMogen law
$\equiv$	$\sim ((p \vee q) \wedge (r \ \vee q \wedge \sim p))$	DeMogen law
$\equiv$	$\sim ((p \vee q) \wedge (r \vee \sim p))$	equivalence property

The last step is optional. The diagram is omitted.

5. Proof Method (20)
Prove formally that if $\sqrt{10}$ is rational, so is $\sqrt{6}$.

Answer: The sentence is an implication and we will prove that the premise is false, i.e., $\sqrt{10}$ is irrational. Once the premise is false, the implication is always true.

We prove that $\sqrt{10}$ is irrational by contradiction. Suppose $\sqrt{10}$ is rational, then there exist two integers a, b, where $gcd(a, b) = 1$ and $\sqrt{10} = a/b$. Hence $10 = a^2/b^2$ or $a^2 = 10b^2 = 2(5b^2)$. Hence a is even and let $a = 2c$ for some integer c. From $a^2 = 10b^2$ we have $(2c)^2 = 10b^2$ or $2c^2 = 5b^2$. That shows that b is also even. Both a and b being even is a contradiction to $gcd(a, b) = 1$. So $\sqrt{10}$ cannot be rational.