22c:131 Limits of Computation

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7.5 Additional NP-Complete Problems

How to Prove NP-Completeness

- A problem B is NP-complete if
  - (membership) $B \in NP$
  - (NP-hard) For all $A \in NP$, $A \leq_P B$
- **Theorems**
  - *SAT* is NP-complete.
  - If B is **NP**-complete and $B \leq_P C$ then C is **NP**-hard.
  - The 3SAT problem is polynomial time reducible to CLIQUE.

Other NP-complete Languages

To show that C is NP-complete, we provide a polynomial time reduction from $3SAT$ to C.

- *VERTEX-COVER* = $\{ \langle G, k \rangle \mid G$ is an undicted graph that has a k-node vertex cover $\}$.

VERTEX-COVER is NP-complete

- VERTEX-COVER is in NP.
- VERTEX-COVER is in NP-hard.
  - Proof1: Using graph properties.
  - Proof2: Reduce 3SAT to VERTEX-COVER.

Proof 2: VERTEX-COVER is NP-hard

Theorem 7.44 3SAT $\leq_P$ VERTEX-COVER.

- $f(\Phi) = \langle G, m + 2n \rangle$, where Φ is a set of n clauses on m variables.
- $G = (V, E)$, where $V = \{x_i, \bar{x}_i \mid 1 \leq i \leq m\} \cup \{l_{i,j} \mid 1 \leq i \leq 3, 1 \leq j \leq n\}$ and
- $E = \{(l_{1,j}, l_{2,j}), (l_{1,j}, l_{3,j}), (l_{2,j}, l_{3,j}) \mid 1 \leq j \leq l\} \cup \{(x_i, l_{j,k}) \mid 1 \leq i \leq m, 1 \leq j \leq 3, 1 \leq k \leq n,$ the j^{th} literal of clause k is $x_i\}$.

Proof 1: VERTEX-COVER is NP-hard

Given a graph $G = (V, E)$, for any $X \in V$,

- X is a vertex-cover iff $V - X$ is an independent set of G.
- X is a clique of G iff X is an independent set of $\bar{G}$, which is the complement of G.

NP-hardness proof:

- CLIQUE $\leq_P$ INDEPENDENT: $f(\langle G, k \rangle) = \langle \bar{G}, k \rangle$.
- INDEPENDENT $\leq_P$ VERTEX-COVER: $f(\langle G, k \rangle) = \langle G, |V| - k \rangle$.

NP Completeness Proofs

Theorem 7.46 HAM-PATH is NP-complete.

To show that C is NP-complete, we can provide a polynomial time reduction from $3SAT$ to C.

Theorem 7.55 UHAM-PATH is NP-complete.

Other NP-complete Languages

Suppose G is a directed graph:

- HAM-PATH = $\{\langle G, s, t \rangle \mid G$ has a Hamiltonian path from s to t $\}$.
- HAM-PATH0 = $\{\langle G \rangle \mid G$ has a Hamiltonian path $\}$.
- HAM-CYCLE = $\{\langle G \rangle \mid G$ has a Hamiltonian cycle $\}$.

G can be also an undirected graph (UHAM-PATH, UHAM-PATH0, UHAM-CYLCE).

SUBSET-SUM is NP-hard

Theorem 7.56 3SAT $\leq_P$ SUBSET-SUM.
$f(\Phi) = \langle X, t \rangle$, where Φ is a set of n clauses on m variables, X contains $2(m+n)$ numbers of upto $m+k$ digits, and t is a $(m+n)$-digit number whose first m digits are 1's and the rest are 3's.

SUBSET-SUM

- *SUBSET-SUM* = $\{ \langle S, t \rangle \mid S = \{x_1, ..., x_k\}$ and for some $\{y_1, ..., y_t\} \subseteq \{x_1, ..., x_k\}$ we have $\Sigma y_i = t \}$.