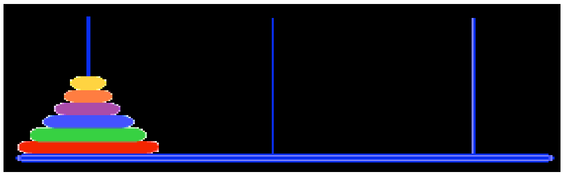
Tower of Hanoi

According to legend, at the dawn of creation in a monastery far away, 64 golden disks of decreasing size were placed on one of three diamond spires with the largest disk on the bottom and progressively smaller disks stacked above it.



The monks of the temple were directed to transfer the disks from one spire to another according to the following rules:
  (i) the disks are to be moved one-by-one off one spire onto another,
  (ii) a larger disk is never to be placed on top of a smaller disk.

When the transfer is completed and all the disks have been moved onto a second spire, the universe will cease to exist.

Naturally we wish to know how many moves are required. Here is a recursive (or inductive, depending on how you look at it) technique for performing the transfer:
  A. suppose we know the most efficient way to move $k \geq 1$ disks from one spire to another
  B. then the most efficient way to transfer k+1 disks from spire-1 to spire-2 is
    1. transfer the top k disks from spire-1 to spire-3
    2. transfer the $(k+1)^{st}$ disk from spire-1 to spire-2
    3. transfer the k disks from spire-3 to spire-2.

Suppose that m_k is the minimum number of moves to transfer k disks. Then
  $m_{k+1} = 1 + 2*m_k$,
and of course, $m_1 = 1$.

Now we wish to "solve" this recurrence to obtain an explicit expression for m_k. We note that the difference equation for m_k is □$m_k = m_{k+1} - m_k = 1 + 2*m_k - m_k = m_k + 1$. Since the difference operator on m_k results again in (essentially) m_k, this suggest that m_k takes the form of an exponential. Hence we assume that $m_k = 2^{k+b} + c$. To determine the constants b and c, we use the various equivalencies. First of all, by the assumed solution $m_{k+1} = 2^{k+1+b} + c$, and by the recurrence $m_{k+1} = 1 + 2m_k = 1 + 2(2^{k+b} + c) = 1 + 2^{k+b+1} + 2c$. Hence $2^{k+1+b} + c = 1 + 2^{k+b+1} + 2c$, and so $c = -1$. Then by the assumed solution $m_1 = 2^{1+b} - 1$ and since $m_1 = 1$, $b = 0$. Therefore the desired solution is

$$m_k = 2^k - 1 \text{ for all } k \geq 1.$$

Since the monks begin with 64 disks, the transfer must require $2^{64} - 1$ moves. For purposes of estimation, we will assume that the monks make one move each second, and that they never make any mistaken false moves. From this calculation we determine that the universe will exist for $2^{64} - 1 =$ 18446744073709551615 seconds, or 5.849424173551e+11 years — approximately 585 billion years. Since the estimated age of the universe is between 10 and 20 billion years, the monks still have a long way to go.