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STRUCTURE PRESERVING PROPERTIES OF CERTAIN CLASSES OF FUNCTIONS ON AUTOMATA

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INTRODUCTION

The effort to analyze computers has given rise to many ideas for a theoretical model of study. In terms of the amount of study given to such models, A. M. Turing's [1] has probably been the most productive. However, in the last few years the idea of a finite automaton has appeared more and more frequently in the literature. This is a model of a machine which accepts only tapes of finite length and which has only a finite number of (internal) states. The motivation of the study of finite automata is provided by the assumption that such a model is more realistic than, for instance, Turing's model.

In many of the studies of finite automata which have appeared, a method for transforming a given automaton into another automaton is introduced. We might cite as examples: reducing a machine to a machine with a minimal number of states [2], [3], raising a machine to a power [4], and forming the direct product of two automata [5].

This paper is devoted to a discussion of structure of automata. In particular, the problem of which structure properties are preserved under certain classes of transformations on automata is studied. It should be noted that, for the results of this paper, the set of states was not assumed to be finite. However, the results of this paper hold for finite automata and a few remarks are made concerning the application of the results to particular transformations.

The material following this introduction proceeds in four sections. In the first section the assumed definition of an automaton is dealt with. This definition is similar to that of Rabin and Scott [5], but is more general in several respects. The remainder of this section defines several structures on automata and explores their interrelations. It is also shown that a topology is naturally established on the set of states of each automaton.

The second section defines what is meant by a function on automata and, in particular, a continuous function. The structure preserving qualities of continuous functions are then investigated and are found to be extremely desirable.

The third section defines and studies a type of function called "operation preserving". It is shown that such functions have an even stronger structure preserving nature than continuous functions. It is also shown that a group, consisting of a particular set of functions, can be associated with each automaton.

The material of the last section is suggested by the result of the previous section which associates a group with each automaton. In particular, the results of this section lead to an answer for the question: when can the elements of the group associated with an automaton be expressed in terms of the next state function of the automaton?

SECTION 1

STRUCTURES AND A TOPOLOGY ON THE SET OF STATES

The definition of an automaton taken here parallels that of Rabin and Scott [5]. Occasionally a weighted, directed graph (state or transition diagram) will be used but only to specify an example. The explanation of this device is delayed until that time.

Definition 1.1 - An automaton, $A = (S, I, M, f)$, is a quadruple where S is a non-empty set (the set of states), I is a non-empty set (the set of inputs), M is a function (the next state function) taking $S \times I$ (Cartesian product) into S, f is a function (input composition) taking $I \times I$ into I such that (I, f) is a semi-group.

Definition 1.1 differs from the usual definition in several respects: first, the set of states is not assumed to be finite; second, the entire set of inputs is included directly in the quadruple; third, the input composition is arbitrary whereas it is usually assumed to be juxtaposition; lastly, an initial state and a set of final states are not specified since in the study of structure this is inessential information.

We now examine some structure properties of automata. Many of the structures defined below are discussed briefly in the literature but in most cases the properties have never been formally set down and their interrelations examined.

Definition 1.2 - A set of states, $T \subset S$, of an automaton $A = (S, I, M, f)$ is open if given any $s \in T$ and any $x \in I$, $M(s, x) \in T$.

Such a set is defined elsewhere in the literature as a stable set [6] or a submachine, but the term "open" is used here due to the topological nature and interpretation of the definitions and results to follow.

Theorem 1.1 - The union of arbitrarily many open sets of states of an automaton $A = (S, I, M, f)$ is an open set of states of A.

Proof: Set $U = \bigcup_K V_\alpha$ where $V_\alpha \subset S$ and V_α is open for all $\alpha \in K$ (index set). Then for $s \in U$, $s \in V_\alpha$ for some $\alpha \in K$. Then since V_α is open, $M(s,x) \in V_\alpha$ for all $x \in I$. Thus $M(s,x) \in U$ for all $x \in I$ and so U is open.

Theorem 1.2 - The intersection of arbitrarily many open sets of states of an automaton $A = (S, I, M, f)$ is an open set of states of A.

Proof: Set $U = \bigcap_K V_\alpha$ where $V_\alpha \subset S$ and V_α is open for all $\alpha \in K$ (index set). For any $s \in U$, $s \in V_\alpha$ for all $\alpha \in K$. Then since V_α is open for each $\alpha \in K$, $M(s,x) \in V_\alpha$ for all $x \in I$ and all $\alpha \in K$. Then $M(s,x) \in U$ for all $x \in I$ and U is open.

Theorem 1.3 - For any automaton $A = (S, I, M, f)$, the collection of open sets of states of A yields a topology on S, the set of states.

Proof: Obviously the null set, $\emptyset$, and the set S are open. This together with Theorems 1.1 and 1.2 establishes a topology [7].

Theorem 1.3 establishes all the structure results which hold for general topological spaces for automata in terms of the open sets of Definition 1.2. Thus we could, for example, follow the topological definition of limit state (point) and closed set and the usual well-known results would already be established.

Definition 1.3 - An automaton $A = (S, I, M, f)$ is <u>sequential</u> if $M(s, f(x, y)) = M(M(s,x), y)$ for all $s \in S$ and $x, y \in I$.

It should be noted that under the definition of an automaton by Rabin and Scott [5] (and similiar considerations due to Moore [3], Mealy [2], etc.) where the input composition is taken to be juxtaposition, the next state function is usually defined on a set of generators and then extended to the entire semi-group by means of the relation in Definition 1.3. Thus a "finite automaton" is usually considered to be sequential by definition. Definition 1.3 deserves one more comment. It will be seen that not only is sequentialness a natural concept, but for many of the results of this section and the next it is indeed necessary.

Definition 1.4 - An automaton $A = (S, I, M, f)$ is <u>strongly connected</u> if given any $s_1, s_2 \in S$, there exists an $x \in I$ such that $M(s_1, x) = s_2$.

The concept of strongly connectedness was first defined and investigated by Moore [3].

Theorem 1.4 - If an automaton $A = (S, I, M, f)$ is strongly connected, then there is no proper open subset of S.

Proof: Assume $U \subset S$ is a proper open subset. Then $S - U \neq \phi$. If $s_1 \in U$ and $s_2 \in S-U$, then $M(s_1, x) \in U$ for all $x \in I$ since U is open. But $s_2 \notin U$, hence $M(s_1, x) \neq s_2$ for all $x \in I$. Thus A is not strongly connected, a contradiction. Hence there is no proper open subset of S.

Lemma 1.1 - If an automaton $A = (S, I, M, f)$ is sequential, then for each $s \in S$, $T_s = \left\{ s_1 \middle| s_1 \in S, M(s,x) = s_1 \right\}$ (i.e., the set of all s_1 such that $M(s,x) = s_1$ for some $x \in I$) is an open set.

Proof: Assume T_s is not open. Then there exists $s_1 \in T_s$ and $x \in I$ such that $M(s_1, x) = s_2 \notin T_s$. Now since $s_1 \in T_s$, $s_1 = M(s, y)$.