# Equivalence Relations

*Definition*: for sets S and T, the **Cartesian product** $S \times T$ is just the set of all *ordered* pairs, $S \times T = \{\langle s,t \rangle \mid s \in S \text{ and } t \in T\}$.

*Definition*: a **binary relation** $\rho$ on set S is a subset $\rho \subseteq S \times S$. For a pair $\langle x,y \rangle \in \rho$, we may alternatively write $x \, \rho \, y$.

*Definition*: a binary relation $\rho$ on set S is an **equivalence relation** provided
(1) $x \, \rho \, x$ for all $x \in S$, (**reflexive** property)
(2) if $x \, \rho \, y$, then $y \, \rho \, x$ (**symmetric** property), and
(3) if $x \, \rho \, y$ and $y \, \rho \, z$, then $x \, \rho \, z$ (**transitive** property).

For each set S, there are two "extreme" equivalence relations — the *identity* relation $I_S = \{\langle x,x \rangle \mid x \in S\}$, and the *universal* relation $U_S = S \times S$. With the identity relation each element is equivalent to only itself, and with the universal relation each element is equivalent to every other.

*Definition*: for an equivalence relation $\rho$ on set S and $x \in S$, the **equivalence class** of x is $[x]_\rho = \{y \in S \mid x \, \rho \, y\}$; if the equivalence relation $\rho$ is understood from context, we may just write $[x]$.

For the identity relation there is a distinct equivalence class for each element containing only that one element. For the universal relation, there is one equivalence class containing all elements.

*Definition*: a collection (finite or infinite) of non-empty subsets of set S, $S_1, S_2, \ldots \subseteq S$ is a **partition** of S provided that:

(1) $S = \bigcup_k S_k$ (**exhaustive**),

(2) $S_i \cap S_j = \varnothing$ if $i \neq j$ (**mutually exclusive**).

The subsets $S_i$ are called the **blocks** of the partition.

*Assertion*: if ρ is an equivalence relation on set S, then the equivalence classes under ρ form the blocks of a partition of S; conversely, for any partition on S, there is an equivalence relation on S whose equivalence classes are the blocks of the partition.

*Definition*: An equivalence relation ρ on set S is a **congruence** for function f: $S \to S$ if for each $s,s' \in S$ so that $s \,\rho\, s'$, $f(s) \,\rho\, f(s')$.

For a congruence, every pair of elements in a common equivalence class is mapped by f to another pair of elements in a common equivalence class, or more briefly, the function preserves the equivalence. In an analogous way we may also speak of an equivalence being a congruence for a function that takes several arguments — whenever each argument is replaced by an equivalent element, the operation produces a result equivalent to the result with the original arguments. And in one last extension, we may speak of an equivalence being a congruence with respect to a *collection* of functions, meaning it is a congruence for each of the functions in the collection.